The center value for the Godunov scheme is (supposing Riemann ICs at $x_{j+1/2} = 0$)

$$u^*(x_{j+1/2}, t) = \begin{cases} \begin{cases} u_l & 0 < s, \\ u_r & 0 > s, \end{cases} & u_l > u_r, \\ \begin{cases} u_l & 0 < f'(u_l), \\ (f')^{-1}(0) & f'(u_l) < 0 < f'(u_r) \\ u_r & f'(u_r) < 0, \end{cases} & u_l < u_r \end{cases}$$

with

$$s := \frac{f(u_r) - f(u_l)}{u_r - u_l}$$

according to my own calculation. We can simplify some more, to get

$$\begin{cases} \begin{cases} u_l & f(u_r) < f(u_l), \\ u_r & f(u_r) > f(u_l), \end{cases} & u_l > u_r, \\ \begin{cases} u_l & 0 < f'(u_l), \\ (f')^{-1}(0) & f'(u_l) < 0 < f'(u_r) \\ u_r & f'(u_r) < 0, \end{cases} & u_l < u_r. \end{cases}$$

This yields

$$\begin{aligned} \hat{f}_{j+1/2} &= f(u^*(x_{j+1/2})) \\ &= \begin{cases} \begin{cases} f(u_l) & f(u_r) < f(u_l), \\ f(u_r) & f(u_r) > f(u_l), \end{cases} & u_l > u_r, \\ \begin{cases} f(u_l) & 0 < f'(u_l), \\ (f')^{-1}(0) & f'(u_l) < 0 < f'(u_r) \\ f(u_r) & f'(u_r) < 0, \end{cases} & u_l < u_r \end{cases} \\ &= \begin{cases} \max_{u \in [u_l, u_r]} f(u) & u_l > u_r, \\ \begin{cases} f(u_l) & 0 < f'(u_l), \\ \min_{u \in [u_l, u_r]} f(u) & f'(u_l) < 0 < f'(u_r) \\ f(u_r) & f'(u_r) < 0, \end{cases} & u_l < u_r \end{cases} \\ &= \begin{cases} \max_{u \in [u_l, u_r]} f(u) & u_l > u_r, \\ \min_{u \in [u_l, u_r]} f(u) & u_l < u_r \end{cases} \end{aligned}$$

which is just what's in the notes, thus verifying the claimed value u^* for the Godunov scheme.