PDE 2

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1 Scalar Conservation Laws

$$u_t + (f(u))_x = 0$$

$x\in\mathbb{R}$, $t>0$, typically f convex. $u(x,0)=u_0(x)$ (given). Prototypical example: *Inviscid Burgers Equation*

$$f(u)=\frac{u^2}{2}.$$

Motivation for Burgers Equation. Fluids in 3 dimensions are described by *Navier-Stokes* equations.

$$\begin{aligned} u_t+u\cdot Du &= -Dp+\nu\Delta u\\ \operatorname{div} u &= 0. \end{aligned}$$

Unknown: u: $\mathbb{R}^3\to\mathbb{R}^3$ velocity, p: $\mathbb{R}^3\to\mathbb{R}$ pressure. ν is a parameter called *viscosity*. Get rid of incompressibility and assume u: $\mathbb{R}\to\mathbb{R}$.

$$u_t+u\,u_x=\nu u_{xx}.$$

Burgers equation (1940s): small correction matters only when u_x is large (Prantl). Method of characteristics:

$$u_t+\left(\frac{u^2}{2}\right)_x=0.$$

Same as $u_t+u\,u_x=0$ if u is smooth. We know how to solve $u_t+c\,u_x=0$. ($c\in\mathbb{R}$ constant) (*1D transport equation*). Assume

$$u=u(x(t),t)$$

By the chain rule

$$\frac{\mathrm{d}u}{\mathrm{d}t}=u_x\frac{\mathrm{d}x}{\mathrm{d}t}+u_t.$$

If $\mathrm{d}x/\mathrm{d}t=u$, we have $\mathrm{d}u/\mathrm{d}t=u\,u_x+u_t=0$. More precisely,

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= 0 \quad \text{along paths}\\ \frac{\mathrm{d}x}{\mathrm{d}t} &= u(x(t),t)=u_0(x(0)). \end{aligned}$$

Suppose $u_0(x)$ is something like this:

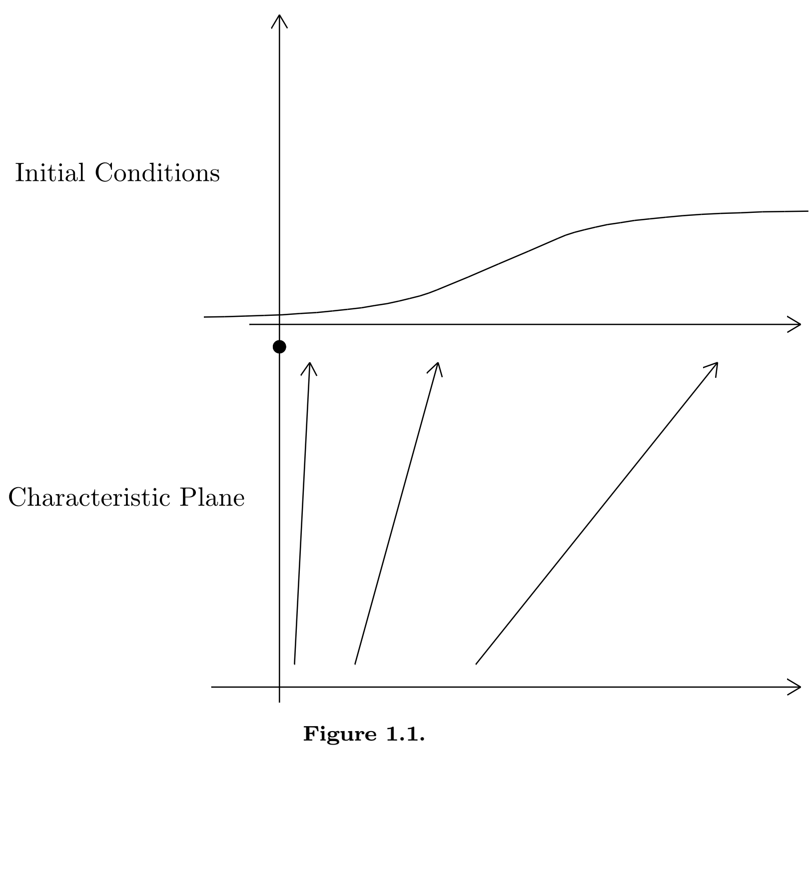

Figure 1.1.

Analytically, $u(x,t)=u_0(x_0)$, $\mathrm{d}x/\mathrm{d}t=u_0(x_0)\Rightarrow x(t)=x(0)+t\,u_0(x_0)$. Strictly speaking, (x,t) is fixed, need to determine x_0. Need to invert $x=x_0+t\,u_0(x_0)$ to find x_0 and thus $u(x,t)=u_0(x_0)$.

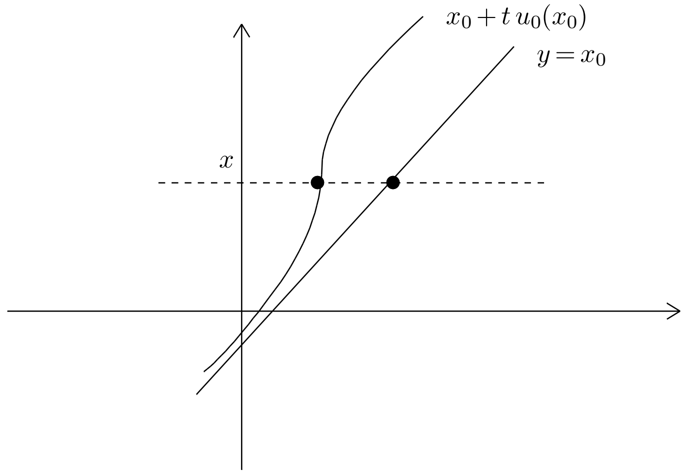

Figure 1.2.

As long as $x_0+t\,u_0(x_0)$ is increasing, this method works. Example 2:

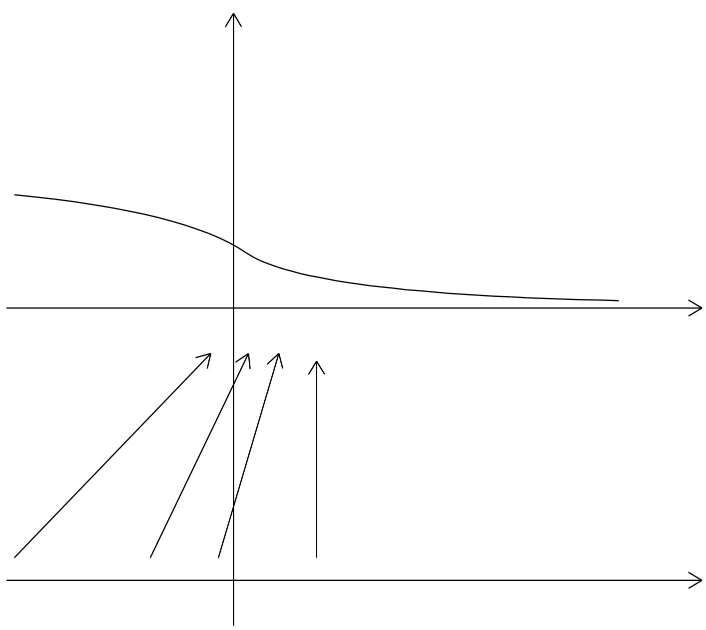

Figure 1.3.

This results in a sort-of breaking wave phenomenon. Analytically, the solution method breaks down when

$$0=\frac{\mathrm{d}x}{\mathrm{d}x_0}=1+t\,u_0'(x_0).$$

No classical (smooth) solutions for all $t > 0$. Let's try weak solutions then. Look for solutions in $\mathcal{D}'$. Pick any test function $f \in C_c^\infty(\mathbb{R} \times [0, \infty))$:

$$\int_0^\infty \int_{\mathbb{R}} \varphi\left[u_t + \left(\frac{u^2}{2}\right)_x\right] = 0, \quad u(x, 0) = u_0(x).$$

Integrate by parts:

$$\int_0^\infty \int_{\mathbb{R}} \left[\varphi_t u + \varphi_x \frac{u^2}{2}\right] \mathrm{d}x\, \mathrm{d}t + \int_{\mathbb{R}} \varphi(x, 0) u_0(x) \mathrm{d}x = 0. \tag{1.1}$$

Definition 1.1. $u \in L^1_{\mathrm{loc}}([0, \infty] \times \mathbb{R})$ *is a weak solution if (1.1) holds for all* $\varphi \in C_c^1([0, \infty) \times \mathbb{R})$.

1.1 Shocks and the Rankine-Hugoniot condition

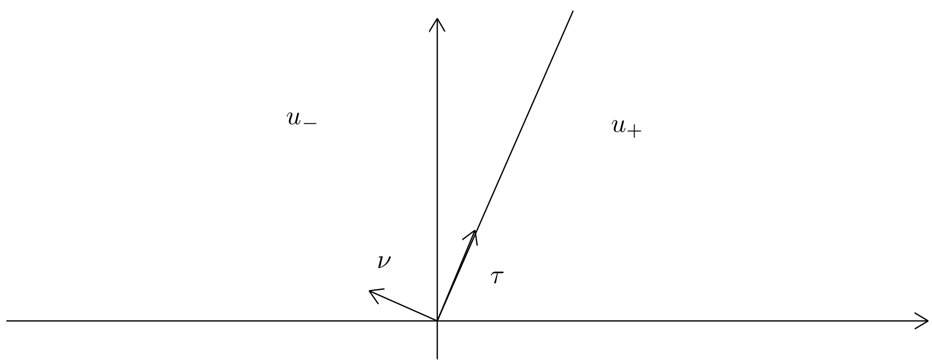

Figure 1.4. Solution for a simple discontinuity (ν and τ are unit vectors.)

Let φ have compact support in $\mathbb{R} \times (0, \infty)$ which crosses the the line of discontinuity. Apply (1.1). Ω_- is the part of the support of φ to the left of the line of discontinuity, Ω_+ the one to the right.

$$\begin{aligned} 0 &= \int_{\Omega_-} \varphi_t u_- + \varphi_x \left(\frac{u_-^2}{2}\right) \mathrm{d}x\, \mathrm{d}t + \int_{\Omega_+} \varphi_t u_+ + \varphi_x \left(\frac{u_+^2}{2}\right) \mathrm{d}x\, \mathrm{d}t \\ &= \int_{\Omega_-} (\varphi u_-)_t + \left(\varphi \frac{u_-^2}{2}\right)_t \mathrm{d}x\, \mathrm{d}t + \cdots \\ &= -\int_\Gamma \varphi\left[u_- \nu_t + \left(\frac{u_-^2}{2}\right)\nu_x\right] \mathrm{d}s + \int_\Gamma \varphi\left[u_+ \nu_t + \left(\frac{u_+^2}{2}\right)\nu_x\right] \mathrm{d}s \end{aligned}$$

Notation $[\![g]\!] = g_+ - g_-$ for any function that jumps across discontinuity. Thus, we have the integrated jump condition

$$\int_\Gamma \varphi\left[[\![u]\!]\nu_t + \left[\!\!\left[\frac{u^2}{2}\right]\!\!\right]\nu_x\right] \mathrm{d}s.$$

Since φ is arbitrary,

$$[u]\nu_t + \left[\!\!\left[\frac{u^2}{2}\right]\!\!\right]\nu_x = 0.$$

For this path,

$$\tau = (\dot{x}, 1) \frac{1}{\sqrt{\dot{x}^2 + 1}}, \quad \nu = (-1, \dot{x}) \frac{1}{\sqrt{\dot{x}^2 + 1}}.$$

($\dot{x}$ is the speed of the shock.)

$$\Rightarrow \dot{x} = \frac{\left[\!\!\left[\frac{u^2}{2}\right]\!\!\right]}{[\![u]\!]} = \frac{u_- + u_+}{2}.$$

Rankine-Hugoniot condition:

$$\text{shock speed} = \frac{[\![f(u)]\!]}{[\![u]\!]}$$

for a scalar conservation law $u_t + (f(u))_x = 0$.

Definition 1.2. *The* Riemann problem *for a scalar conservation law is given by*

$$u_t + (f(u))_x = 0,$$

$$u_0(x) = \begin{cases} u_- & x < 0, \\ u_+ & x \geqslant 0. \end{cases}$$

Example 1.3. Let's consider the Riemann problem for the Burgers equation: $f(u) = u^2/2$.

$$u_0(x) = \begin{cases} 0 & x < 0, \\ 1 & x \geqslant 0. \end{cases}$$

By the derivation for "increasing" initial data above, we obtain

$$u(x,t) = \mathbf{1}_{\{x \geqslant y(t)\}}, \quad y(t) = \frac{\llbracket u^2/2 \rrbracket}{\llbracket u \rrbracket} = \frac{t}{2}.$$

The same initial data admits another (weak) solution. Use characteristics:

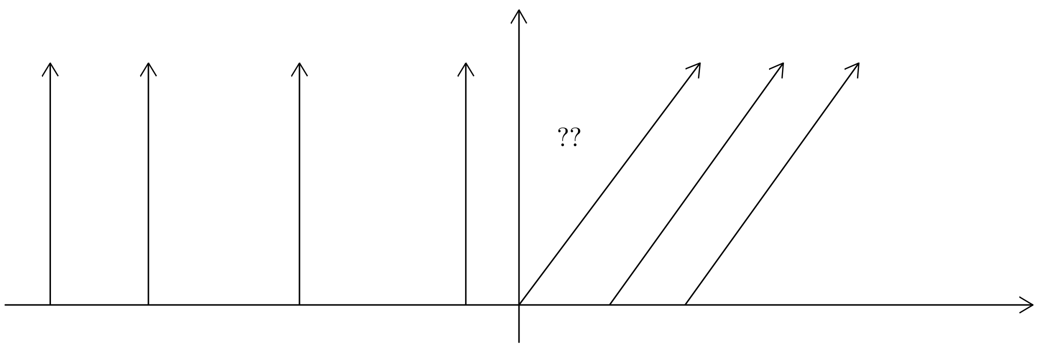

Figure 1.5.

Rarefaction wave: Assume $u(x,t) = v(x/t) =: v(\xi)$. Then

$$\begin{aligned} u_t &= v'\Big(-\frac{x}{t^2}\Big) = \frac{-\xi v'}{t}, \\ u_x &= v'\Big(\frac{1}{t}\Big) = \frac{1}{t} v'. \end{aligned}$$

So, $u_t + u\, u_x = 0 \Rightarrow -\xi/t\, v' + v/t\, v' = 0 \Rightarrow v'(-\xi + v) = 0$. Choose $v(\xi) - \xi$. Then

$$u(x,t) = \frac{x}{t}.$$

Thus we have a second weak solution

$$u(x,t) = \begin{cases} 0 & x < 0, \\ x/t & 0 \leqslant \frac{x}{t} \leqslant 1, \\ 1 & \frac{x}{t} > 1. \end{cases}$$

So, which if any is the *correct* solution? Resolution:

- $f(u) = u^2/2$: E. Hopf, 1950
- General convex f: Lax, Oleinik, 1955.
- Scalar equation in $\mathbb{R}^n$: Kružkov.

1.2 Hopf's treatment of Burgers equation

Basic idea: The "correct" solution to

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

must be determined through a limit as $\varepsilon \searrow 0$ of the solution u^ε of

$$u_t^\varepsilon + u^\varepsilon u_x^\varepsilon = \varepsilon u_{xx}^\varepsilon.$$

This is also called to the *vanishing viscosity method*. Then, apply a clever change of variables. Assume u has compact support. Let

$$U(x,t) = \int_{-\infty}^x u(y,t) \mathrm{d}y.$$

(Hold $\varepsilon > 0$ fixed, drop superscript.)

$$U_t = \int_{-\infty}^x u_t(y,t)\mathrm{d}y = -\int_{-\infty}^x \left(\frac{u^2}{2}\right)_y \mathrm{d}y + \varepsilon \int_{-\infty}^x u_{yy}(y,t)\mathrm{d}y.$$

Then

$$U_t = -\frac{u^2}{2} + \varepsilon u_x$$

or

$$U_t + \frac{U_x^2}{2} = \varepsilon U_{xx}. \tag{1.2}$$

Equations of the form $U_t + H(Du) = 0$ are called *Hamilton-Jacobi equations*. Let

$$\psi(x,t) = \exp\left(-\frac{U(x,t)}{2\varepsilon}\right)$$

(Cole-Hopf)

$$\begin{aligned}
\psi_t &= \psi\left(-\frac{1}{2\varepsilon}U_t\right)\\
\psi_x &= \psi\left(-\frac{1}{2\varepsilon}U_x\right)\\
\psi_{xx} &= \psi\left(-\frac{1}{2\varepsilon}U_x\right)^2 + \psi\left(-\frac{1}{2\varepsilon}U_{xx}\right).
\end{aligned}$$

Use (1.2) to see that

$$\psi_t = \varepsilon \psi_{xx},$$

which is the heat equation for $x \in \mathbb{R}$, and

$$\psi_0(x) = \exp\left(-\frac{U_0(x)}{2\varepsilon}\right).$$

Since $\psi > 0$, uniqueness by Widder.

$$\psi(x,t) = \frac{1}{\sqrt{4\pi t\varepsilon}} \int_{\mathbb{R}} \exp\left(-\frac{1}{2\varepsilon}\left[\frac{(x-y)^2}{2t} + U_0(y)\right]\right)\mathrm{d}y.$$

Define

$$G(t,x,y) = \frac{(x-y)^2}{2t} + U_0(y),$$

which is called the *Cole-Hopf* function. Finally, recover $u(x,t)$ via

$$\begin{aligned}
u(x,t) = -2\varepsilon\psi_x/\psi &= -2\varepsilon \frac{\int_{\mathbb{R}} \frac{-2(x-y)}{2\varepsilon\, 2t} \exp\left(-\frac{G}{2\varepsilon}\right)\mathrm{d}y}{\int_{\mathbb{R}} \exp\left(-\frac{G}{2\varepsilon}\right)\mathrm{d}y} = \frac{\int_{\mathbb{R}} \frac{x-y}{t} \exp\left(-\frac{G}{2\varepsilon}\right)\mathrm{d}y}{\int_{\mathbb{R}} \exp\left(-\frac{G}{2\varepsilon}\right)\mathrm{d}y}\\
&= \frac{x}{t} - \frac{1}{t} \cdot \frac{\int_{\mathbb{R}} y \exp\left(-\frac{G}{2\varepsilon}\right)\mathrm{d}y}{\int_{\mathbb{R}} \exp\left(-\frac{G}{2\varepsilon}\right)\mathrm{d}y}.
\end{aligned}$$

Heuristics: We want $\lim_{\varepsilon \to 0} u^{\varepsilon}(x,t)$.

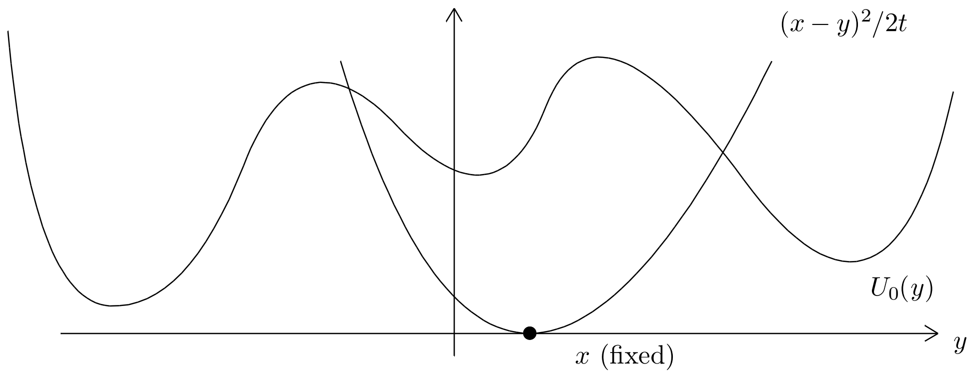

Figure 1.6.

Add to get $G(x,y,t)$. We hold x,t fixed and consider $\varepsilon \downarrow 0$. Let $a(x,t)$ be the point where $G=0$. We'd expect

$$\lim_{\varepsilon \to 0} u^{\varepsilon}(x,t) = \frac{x - a(x,t)}{t}.$$

Problems:

- G may not have a unique minimum.
- G need not be C^2 near minimum.

Assumptions:

- U_0 is continuous (could be weakened)
- $U_0(y) = o(|y|^2)$ as $|x| \to \infty$.

Definition 1.4. [The inverse Lagrangian function]

$$\begin{aligned} a_-(x,t) &= \inf \left\{ z \in \mathbb{R} \colon G(x,z,t) = \min_y G \right\} = \inf \operatorname{argmin} G, \\ a_+(x,t) &= \sup \left\{ z \in \mathbb{R} \colon G(x,z,t) = \min_y G \right\} = \sup \operatorname{argmin} G, \end{aligned}$$

Lemma 1.5. *Use our two basic assumptions from above. Then*

- *These functions are well-defined.*
- $a_+(x_1,t) \leqslant a_-(x_2,t)$ *for* $x_1 < x_2$. *In particular,* a_-, a_+ *are increasing (non-decreasing).*
- a_- *is left-continuous,* a_+ *is right-continuous:* $a_+(x,t) = a_+(x_+,t)$.
- $\lim_{x \to \infty} a_-(x,t) = +\infty$, $\lim_{x \to -\infty} a_+(x,t) = -\infty$.

In particular, $a_+ = a_-$ *except for a countable set of points* $x \in \mathbb{R}$ *(These are called* shocks*).*

Theorem 1.6. (Hopf) *Use our two basic assumptions from above. Then for every* $x \in \mathbb{R}$, $t > 0$

$$\frac{x - a_+(x,t)}{t} \leqslant \liminf_{\varepsilon \to 0} u^{\varepsilon}(x,t) \leqslant \limsup_{\varepsilon \to 0} u^{\varepsilon}(x,t) \leqslant \frac{x - a_-(x,t)}{t}.$$

In particular, for every $t > 0$ *except for* x *in a countable set, we have*

$$\lim_{\varepsilon \to 0} u^{\varepsilon}(x,t) = \frac{x - a_+(x,t)}{t} = \frac{x - a_-(x,t)}{t}.$$

Graphical solution I (Burgers): Treat $U_0(y)$ as given.

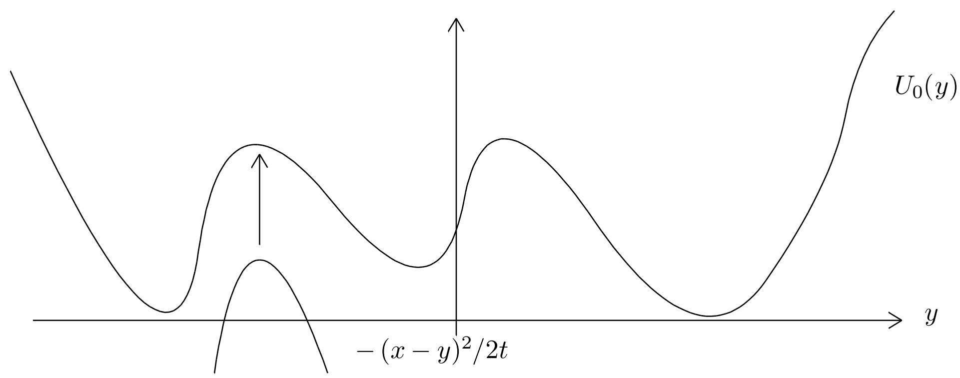

Figure 1.7.

$U_0(y) > C - (x-y)^2/2t$ is parabola is below $U_0(y)$. Then

$$U_0(y) + \frac{(x-y)^2}{2t} - C > 0,$$

where C is chosen so that the two terms "touch".

Graphical solution II: Let

$$H(x,y,t) = G(x,y,t) - \frac{x^2}{2t} = U_0(y) + \frac{(x-y)^2}{2t} - \frac{x^2}{2t} = U_0(y) + \frac{y^2}{2t} - \frac{x\,y}{t}.$$

Observe H, G have minima at same points for fixed x, t.

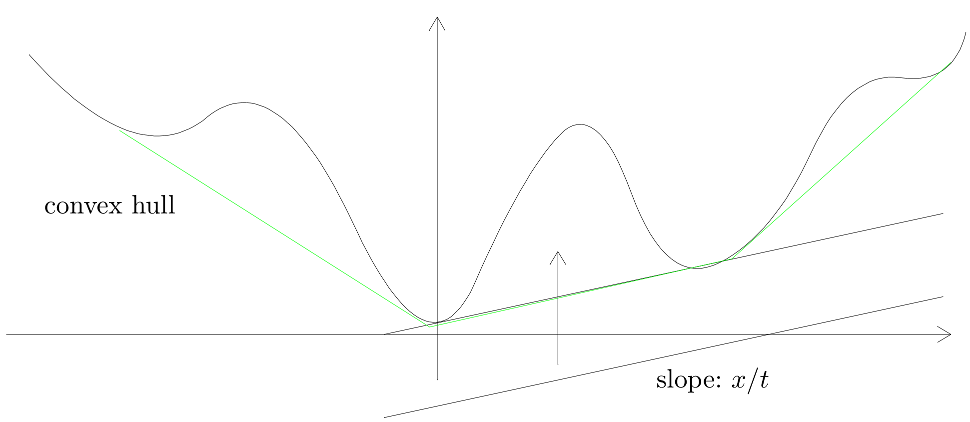

Figure 1.8.

Definition 1.7. *If* $f: \mathbb{R}^n \to \mathbb{R}$ *continuous, then the* convex hull *of* f *is*

$$\sup_g \{f \geqslant g: g \text{ convex}\}.$$

a_+, a_- defined by $U_0(y) + y^2/2t$ same as that obtained from the convex hull of $U_0(y) + y^2/2t \Rightarrow$ Irreversibility.

Remark 1.8. Suppose $U_0 \in C^2$. Observe that at a critical point of G, we have

$$\partial_y G(x, y, t) = 0,$$

which means

$$\partial_y \left[U_0(y) + \frac{(x-y)^2}{2t} \right] = 0,$$

so

$$u_0(y) + \frac{(y-x)}{t} = 0 \Rightarrow x = y + t\, u_0(y).$$

Every y such that $y + t\, u_0(y) = x$ gives a Lagrangian point that arrives at x at the time t.

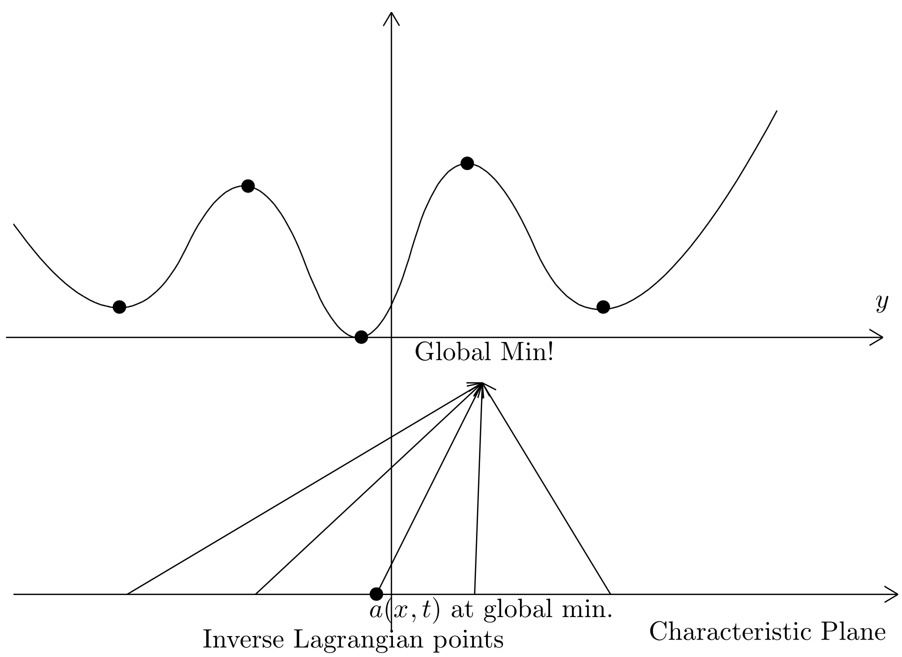

Figure 1.9.

Remark 1.9. The main point of the Cole-Hopf method is that we have a solution formula independent of ε, and thus provides a uniqueness criteria for suitable solutions.

Exact references for source papers are:

- Eberhard Hopf, CPAM 1950 "The PDE $u_t + u\, u_x = \mu u_{xx}$"
- S.N. Kružkov, Math USSR Sbornik, Vol. 10, 1970 #2.

$$S_{(x,t)} = \left\{ z \in \mathbb{R} : G(x, z, t) = \min_y G \right\}$$

Proof. [Lemma 1.5] Observe that $G(x, y, t)$ is continuous in y, and

$$\lim_{|y| \to \infty} \frac{G(x, y, t)}{|y|^2} = \lim_{|y| \to \infty} \frac{(x-y)^2}{2t|y|^2} + \frac{U_0(y)}{|y|^2} = \frac{1}{2t} > 0.$$

Therefore, minima of G exist and $S_{(x,t)}$ is a bounded set for $t > 0$.

$$\begin{aligned} \Rightarrow a_-(x,t) = \inf S_{(x,t)} &> -\infty, \\ a_+(x,t) = \sup S_{(x,t)} &< \infty. \end{aligned}$$

Proof of monotinicity: Fix $x_2 > x_1$. For brevity, let $z = a_+(x_1, t)$. We'll show $G(x_2, y, t) > G(x_2, z, t)$ for any $y < z$. This shows that $\min_y G(x_2, y, t)$ can only be achieved in $[z, \infty)$, which implies $a_-(x_2, t) \geqslant z = a_+(x_1, t)$. Use definition of G:

$$\begin{aligned}
G(x_2, y, t) - G(x_2, z, t) &= \frac{(x-y)^2}{2t} + U_0(y) - \frac{(x_2-z)^2}{2t} - U_0(z) \\
&= \left[\frac{(x_1-y)^2}{2t} + U_0(y)\right] - \left[\frac{(x_1-z)^2}{2t} + U_0(z)\right] + \frac{1}{2t}\big[(x_2-y)^2 - (x_1-y)^2 + (x_1 - z)^2 - (x_2-z)^2\big] \\
&= \underbrace{G(x, y, t) - G(x, z, t)}_{a)} + \frac{1}{t}\Big[\underbrace{(x_2-x_1)(z-y)}_{b)}\Big]
\end{aligned}$$

$a) \geqslant 0$ because $G(x, z, t) = \min G(x, \cdot\,, t)$, $b) > 0$ because $x_2 > x_1$, by assumption $z > y$. By definition, $a_-(x_2, t) \leqslant a_+(x_2, t)$. So in particular,

$$a_+(x_1, t) \leqslant a_+(x_2, t),$$

so a_+ is increasing. Proof of other properties is similar. □

Corollary 1.10. *$a_-(x, t) = a_+(x, t)$ at all but a countable set of points.*

Proof. We know a_-, a_+ are increasing functions and bounded on finite sets. Therefore,

$$\lim_{y \to x_-} a_\pm(y, t), \qquad \lim_{y \to x_+} a_\pm(y, t)$$

exist at all $x \in \mathbb{R}$. Let $F = \{x: a_+(x_-, t) < a_-(x_+, t)\}$. Then F is countable.

Claim: $a_-(x, t) = a_+(x, t)$ for $x \notin F$.

$$a_+(y_1, t) \leqslant a_-(y_2, t) \leqslant a_+(y_3, t).$$

Therefore,

$$\lim_{y \to x} a_-(y, t) = a_+(x, t).$$

□

Remark 1.11. Hopf proves a stronger version of Theorem 1.6:

$$\frac{x - a_+(x, t)}{t} \leqslant \liminf_{\varepsilon \to 0, \xi \to x, \tau \to t} u^\varepsilon(\xi, \tau) \leqslant \limsup_{\varepsilon \to 0, \xi \to x, \tau \to t} u^\varepsilon(\xi, \tau) \leqslant \frac{x - a_-(x, t)}{t}.$$

Proof. (of Theorem 1.6) Use the explicit solution to write

$$u^\varepsilon(x, t) = \frac{\int_{\mathbb{R}} \frac{x-y}{t} \cdot \exp\left(\frac{-P}{2t}\right) \mathrm{d}y}{\int_{\mathbb{R}} \exp\left(\frac{-P}{2t}\right) \mathrm{d}y},$$

where $P(x, y, t) = G(x, y, t) - m(x, t)$ with $m(x, t) = \min_y G$.

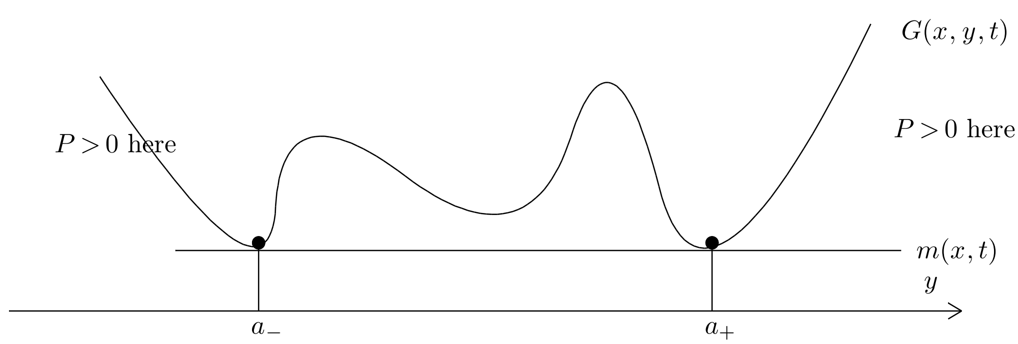

Figure 1.10.

Fix x,t. Fix $\eta>0$, let a_+ and a_- denote $a_+(x,t)$ and $a_-(x,t)$. Let

$$\begin{aligned} l &:= \frac{x-a_+-\eta}{t} \\ &\leqslant \frac{x-a_--\eta}{t} =: L. \end{aligned}$$

Lower estimate

$$\liminf_{\varepsilon\to 0} u^\varepsilon(x,t) \geqslant \frac{x-a_+}{t}-\eta.$$

Consider

$$u^\varepsilon(x,t)-l = \frac{\int_{\mathbb{R}}\left(\frac{x-y}{t}-l\right)\cdot\exp\left(\frac{-P}{2\varepsilon}\right)\mathrm{d}y}{\int_{\mathbb{R}}\exp\left(\frac{-P}{2\varepsilon}\right)\mathrm{d}y} = \frac{\int_{\mathbb{R}}\left(\frac{a_++\eta-y}{t}-l\right)\cdot\exp\left(\frac{-P}{2\varepsilon}\right)\mathrm{d}y}{\int_{\mathbb{R}}\exp\left(\frac{-P}{2\varepsilon}\right)\mathrm{d}y}.$$

Estimate the numerator as follows:

$$\begin{aligned} \int_{-\infty}^{\infty}\frac{a_++\eta-y}{t}\cdot\exp\left(\frac{-P}{2\varepsilon}\right)\mathrm{d}y &= \underbrace{\int_{-\infty}^{a_+}}_{\geqslant 0}+\int_{a_+}^{\infty} \\ \int_{\mathbb{R}} &\geqslant \int_{a_++\eta}^{\infty}\frac{a_++\eta-y}{t}\exp\left(\frac{-P}{2\varepsilon}\right)\mathrm{d}y \end{aligned}$$

On the interval $y\in[a_++\eta,\infty]$, we have the uniform lower bound

$$\frac{P(x,y,t)}{(y-a_+)^2}\geqslant\frac{A}{2}>0$$

for some constant A depending only on η. Here we use

$$\frac{P(x,y,t)}{|y|^2}=\frac{U_0(y)}{|y|^2}+\frac{(x-y)^2}{2t|y|^2}-\frac{m(x,t)}{|y|^2}\to\frac{1}{2t}>0$$

as $|y|\to\infty$. We estimate

$$\begin{aligned} \int_{a_++\eta}^{\infty}\frac{|a_++\eta-y|}{t}e^{-P/2\varepsilon}\mathrm{d}y &\leqslant \int_{a_++\eta}^{\infty}\frac{|a_++\eta-y|}{t}\exp\left(-\frac{A}{4\varepsilon}(y-a_+)^2\right)\mathrm{d}y \\ &= \int_{\eta}^{\infty}\frac{(y-\eta)}{t}\exp\left(-\frac{A\,y^2}{4\varepsilon}\right)\mathrm{d}y \\ &< \int_{\eta}^{\infty}\frac{y}{t}\exp\left(-\frac{A\,y^2}{4\varepsilon}\right)\mathrm{d}y \\ &= \frac{1}{t}\frac{\varepsilon}{A}\int_{\sqrt{\frac{A}{\varepsilon}}\eta}^{\infty}y\,e^{-y^2/2}\mathrm{d}y=\frac{1}{t}\cdot\frac{\varepsilon}{A}e^{-\frac{A\eta^2}{2\varepsilon}}. \end{aligned}$$

For the denominator,

$$\int_{\mathbb{R}}\exp\left(\frac{-P}{2\varepsilon}\right)\mathrm{d}y:$$

Since P is continuous, and $P(x,a_+,t)=0$, there exists δ depending only on η such that

$$P(x,y,t)\leqslant\frac{A}{2}\eta$$

for $y\in[a_+,a_++\delta]$. Thus,

$$\int_{\mathbb{R}}e^{-P/2\varepsilon}\mathrm{d}y\geqslant\int_{a_+}^{a_++\delta}e^{-P/2\varepsilon}\mathrm{d}y\geqslant\int_{a_+}^{a_++\delta}e^{-(A/2\varepsilon)\eta^2}\mathrm{d}y=\delta e^{-(A/2\varepsilon)\eta^2}.$$

Combine our two estimates to obtain

$$u^\varepsilon(x,t)-l\geqslant\frac{-\varepsilon e^{-(A/2\varepsilon)\eta^2}}{A\,t\,\delta e^{-(A/2\varepsilon)\eta^2}}=-\varepsilon\cdot\frac{1}{A\,t\delta}.$$

Since A, δ depend only on η,

$$\liminf_{\varepsilon\to 0} u^{\varepsilon}(x,t) \geqslant l = \frac{x-a_+-\eta}{t}.$$

Since $\eta>0$ arbitrary,

$$\liminf_{\varepsilon\to 0} u^{\varepsilon}(x,t) = \frac{x-a_+}{t}.$$

$\square$

Corollary 1.12. $\lim_{\varepsilon\to 0} u^{\varepsilon}(x,t)$ *exists at all but a countable set of points and defines* $u\in \mathrm{BV}_{\mathrm{loc}}$ *with left and right limits at all* $x\in\mathbb{R}^n$.

Proof. We know

$$a_+(x,t) = a_-(x,t)$$

at all but a countable set of shocks. So,

$$\lim_{\varepsilon\to 0} u^{\varepsilon}(x,t) = \frac{x-a_+(x,t)}{t} = \frac{x-a_-(x,t)}{t}$$

at these points. $\mathrm{BV}_{\mathrm{loc}}$ because we have the difference of increasing functions. $\square$

Corollary 1.13. *Suppose* $u_0\in \mathrm{BC}(\mathbb{R})$ *(bounded, continuous). Then*

$$u(\,\cdot\,,t) = \lim_{\varepsilon\to 0} u^{\varepsilon}(\,\cdot\,,t)$$

is bounded and is a weak solution to

$$u_t + \left(\frac{u^2}{2}\right)_x = 0.$$

Proof. Suppose $\varphi\in C_c^{\infty}(\mathbb{R}\times(0,\infty))$. Then we have

$$\varphi\left(u_t^{\varepsilon} + \left(\frac{u^{\varepsilon}}{2}\right)_x\right) = (\varepsilon u_{xx}^{\varepsilon})\varphi$$

$$\int_0^{\infty}\int_{\mathbb{R}}\left[\varphi_t u^{\varepsilon} + \varphi_x\frac{(u^{\varepsilon})^2}{2}\right]\mathrm{d}x\,\mathrm{d}t = \varepsilon\int_0^{\infty}\int_{\mathbb{R}}\varphi_{xx}u^{\varepsilon}\mathrm{d}x\,\mathrm{d}t.$$

We want

$$-\int_0^{\infty}\left[\varphi_t u + \varphi_x\frac{u^2}{2}\right]\mathrm{d}x\,\mathrm{d}t = 0.$$

Suppose

$$u_t^{\varepsilon} + u^{\varepsilon}u_x^{\varepsilon} = \varepsilon u_{xx}^{\varepsilon}, \quad u^{\varepsilon}(x,0)\in \mathrm{BC}(\mathbb{R}).$$

Maximum principle yields

$$\|u^{\varepsilon}(\,\cdot\,,t)\|_{L^{\infty}} \leqslant \|u_0\|_{L^{\infty}}.$$

Use DCT+$\lim_{\varepsilon\to 0} u^{\varepsilon}(x,t) = u$ a.e. to pass to limit. $\square$

1.3 Two basic examples of Solutions

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

$u(x,0) = u_0(x)$, $U_0(x) = \int_0^x u_0(y)\mathrm{d}y$. Always consider the Cole-Hopf solution.

$$u(x,t) = \frac{x-a(x,t)}{t},$$

$$a(x,t) = \mathrm{argmin}\underbrace{\frac{(x-y)^2}{2t} + U_0(y)}_{G(x,y,t)}.$$

Example 1.14. $u_0(x) = \mathbf{1}_{\{x>0\}}$. Here,

$$U_0(y) = \int_0^y \mathbf{1}_{\{y'>0\}} \mathrm{d}y' = y\mathbf{1}_{\{y>0\}}$$

Then

$$G(x, y, t) = \frac{(x-y)^2}{2t} + y\mathbf{1}_{\{y>0\}} \geqslant 0,$$

and

$$G(x, y, t) = 0 = x\mathbf{1}_{\{x>0\}} = 0 \quad [???]$$

if $x \leqslant 0$. So, $a = x$ for $x \leqslant 0$. Differentiate G and set $= 0$

$$0 = \frac{y-x}{t} + 1 \quad (\text{assuming } y > 0)$$

So, $y = x - t$. Consistency: need $y > 0 \Rightarrow x > t$. Gives $u(x, t) = 1$ for $x > t$.

$$\begin{aligned} G(x, y, t) &= \frac{x^2}{2t} + \frac{y^2}{2t} - \frac{x\,y}{t} + y\mathbf{1}_{\{y>0\}} \\ &= \frac{x^2}{2t} + \frac{y^2}{2t} + y\Big(\mathbf{1}_{\{y>0\}} - \frac{x}{t}\Big). \end{aligned}$$

Consider $0 < x/t < 1$, $t > 0$. *Claim:* $G(x, y, t) \geqslant x^2/2t$ and $a = 0$.

- Case I: $y < 0$, then $G(x, y, t) - x^2/2t = y^2/2t - x\,y/t > 0$.
- Case II: $y > 0$, then $G(x, y, t) - x^2/2t = y^2/2t + (1 - x/t)y > 0$.

$$a(x, t) = \begin{cases} x & x \leqslant 0, \\ 0 & 0 < x \leqslant t, \\ x - t & x \geqslant t. \end{cases}$$

Then

$$u(x, t) = \frac{x - a(x, t)}{t} = \begin{cases} 0 & x \leqslant 0, \\ x/t & 0 < x \leqslant t, \\ 1 & t \leqslant x. \end{cases}$$

Example 1.15. $u_0(x) = -\mathbf{1}_{\{x>0\}}$. Then

$$u(x, t) = -\mathbf{1}_{\{x > -t/2\}}.$$

Shock path: $x = -t/2$.

Here are some properties of the Cole-Hopf solution:

- $u(\,\cdot\,, t) \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}) \;\rightarrow\;$ difference of two increasing functions
- $u(x_-, t)$ and $u(x_+, t)$ exist at all $x \in \mathbb{R}$. And $u(x_-, t) \geqslant u(x_+, t)$. In particular,

  $$u(x_-, t) > u(x_+, t)$$

  at jumps. This is the *Lax-Oleinik entropy condition*. It says that chracteristics always enter a shock, but never leave it.
- Suppose $u(x_-, t) > u(x_+, t)$. We have the Rankine-Hugoniot condidtion:

  $$\text{Velocity of shock} = \frac{[\![\frac{u^2}{2}]\!]}{[\![u]\!]} = \frac{1}{2}(u(x_+, t) + u(x_-, t)).$$

  Claim: If x is a shock location

  $$\frac{1}{2}(u(x_-, t) + u(x_+, t)) = \frac{1}{a(x_+, t) - a(x_-, t)} \int_{a_-}^{a_+} u_0(y)\mathrm{d}y.$$

  $$\underbrace{(a_+ - a_-)(\text{velocity of shock})}_{\text{final momentum}} \;=\; \underbrace{\int_{a_-}^{a_+} u_0(y)\mathrm{d}y}_{\text{initial momentum}}$$

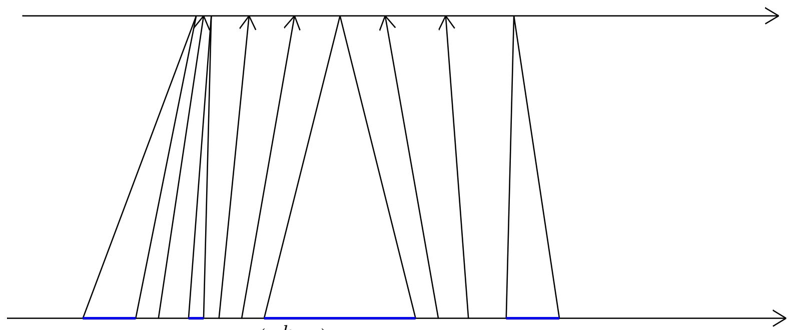

Figure 1.11. The "clustering picture".

1.4 Entropies and Admissibility Criteria

$$\begin{aligned} u_t + D\cdot(f(u)) &= 0\\ u(x,0) &= u_0(x)\end{aligned}$$

for $x \in \mathbb{R}^n$, $t > 0$. Many space dimensions, but u is a scalar $u\colon \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$, $f\colon \mathbb{R} \to \mathbb{R}^n$ (which we assume to be C^1, but which usually is C^∞). Basic calculation: Suppose $u \in C_c^\infty(\mathbb{R}^n \times [0, \infty))$, and also suppose we have a convex function $\eta\colon \mathbb{R} \to \mathbb{R}$ (example: $\eta(u) = u^2/2$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} \eta(u)\mathrm{d}x = \int_{\mathbb{R}^n} \eta'(u)\, u_t \mathrm{d}x = -\int_{\mathbb{R}^n} \eta'(u) D_x(f(u))\mathrm{d}x.$$

Suppose we have a function $q\colon \mathbb{R} \to \mathbb{R}^n$ such that

$$D_x q(u) = \eta'(u) D_x(f(u)),$$

i.e.

$$\begin{aligned} \partial_{x_1} q_1(u) + \partial_{x_2} q_2(u) + \cdots + \partial_{x_n} q_n(u) &= q_1' u_{x_1} + q_2' u_{x_2} + \cdots + q_n' u_{x_n}\\ &\overset{\text{RHS}}{=} \eta'(u) f_1' u_{x_1} + \eta'(u) f_2' u_{x_2} + \cdots + \eta'(u) f_n' u_{x_n}.\end{aligned}$$

Always holds: Simply define $q_i' = \eta'(u) f_i'$. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} \eta(u)\mathrm{d}x = -\int_{\mathbb{R}^n} \operatorname{div} q(u)\mathrm{d}x = -\int_{\text{“}\partial\mathbb{R}^n\text{”}} q(u)\cdot \boldsymbol{n} = 0,$$

provided $q(0) = 0$.

Example 1.16. Suppose $u_t + u\, u_x = 0$. Here $f'(u) = u$. If $\eta(u) = u^2/2$, $q'(u) = \eta'(u) f'(u) = u^2$. So, $q(u) = u^3/3$. Smooth solution to Burgers Equation:

$$\partial_t\left(\frac{u^2}{2}\right) + \partial_x\left(\frac{u^3}{3}\right) = 0.$$

(called the *companion balance law*) And

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{u^2}{2}\mathrm{d}x = 0,$$

which is *conservation of energy*.

Consider what happens if we add viscosity

$$\begin{aligned} u_t^\varepsilon + D_x\cdot(f(u^\varepsilon)) &= \varepsilon \Delta u^\varepsilon,\\ u^\varepsilon(x,0) &= u_0(x).\end{aligned}$$

In this case, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\eta(u^\varepsilon)\mathrm{d}x &= \int_{\mathbb{R}^n}\eta'(u^\varepsilon)u_t^\varepsilon\mathrm{d}x = \underbrace{-\int_{\mathbb{R}^n} D_x\cdot(q(u^\varepsilon))\mathrm{d}x}_{=0} + \varepsilon\int_{\mathbb{R}^n}\eta'(u^\varepsilon)D_x\cdot D_x u_\varepsilon \mathrm{d}x\\ &= -\varepsilon\int_{\mathbb{R}^n}\underbrace{\eta''(u^\varepsilon)}_{\geqslant 0}|Du^\varepsilon|^2\mathrm{d}x < 0\end{aligned}$$

because η is *convex*. If a solution to our original system is $\lim_{\varepsilon\to 0} u^\varepsilon$ of solutions of the viscosity system, we must have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\eta(u)\mathrm{d}x \leqslant 0.$$

Fundamental convex functions (*Kružkov entropies*): $(u-k)_+$, $(k-u)_+$, $|u-k|$.

Definition 1.17. (Kružkov) *A function $u\in L^\infty(\mathbb{R}^n\times(0,\infty))$ is an* entropy *(or* admissible*)* solution *to the original system, provided*

1. *For every $\varphi\in C_c^\infty(\mathbb{R}^n\times(0,\infty))$ with $\varphi\geqslant 0$ and every $k\in\mathbb{R}$ we have*

$$\int_0^\infty\int_{\mathbb{R}}[|u-k|\varphi_t + \operatorname{sgn}(u-k)(f(u)-f(k))\cdot D_x\varphi]\mathrm{d}x\,\mathrm{d}t \geqslant 0. \tag{1.3}$$

2. *There exists a set F of measure zero such that for $t\notin F$, $u(\cdot,t)\in L^\infty(\mathbb{R}^n)$ and for any ball $B(x,r)$*

$$\lim_{t\to 0, t\notin F}\int_{B(x,r)}|u(y,t)-u_0(y)|\mathrm{d}y = 0.$$

An alternative way to state Condition 1 above is as follows: For every (entropy, entropy-flux) pair (η, q), we have

$$\partial_t\eta(u) + \partial_x(q(u)) \leqslant 0 \tag{1.4}$$

in $\mathcal{D}'$. Recover (1.3) by choosing $\eta(u)=|u-k|$. (1.3)$\Rightarrow$(1.4) because all convex η can be generated from the fundamental entropies.

(1.3) means that if we multiply by $\varphi\geqslant 0$ and integrate by parts we have

$$-\int_0^\infty\int_{\mathbb{R}^n}[\varphi_t\eta(u) + D_x\varphi\cdot q(u)]\mathrm{d}x\,\mathrm{d}t \leqslant 0.$$

Positive distributions are measures, so

$$\partial_t\eta(u) + \partial_x(q(u)) = -m_\eta,$$

where m_η is some measure that depends on η. To be concrete, consider Burgers equation and $\eta(u)=u^2/2$ (energy). Dissipation in Burgers equation:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}(u^\varepsilon)^2\mathrm{d}x &= -2\int_{\mathbb{R}}(u^\varepsilon)^2u_x^\varepsilon + 2\varepsilon\int_{\mathbb{R}}u^\varepsilon u_{xx}^\varepsilon\mathrm{d}x\\ &= -2\varepsilon\int_{\mathbb{R}}(u_x^\varepsilon)^2\mathrm{d}x.\end{aligned}$$

But what is the limit of the integral term as $\varepsilon\to 0$? Suppose we have a situation like in the following figure:

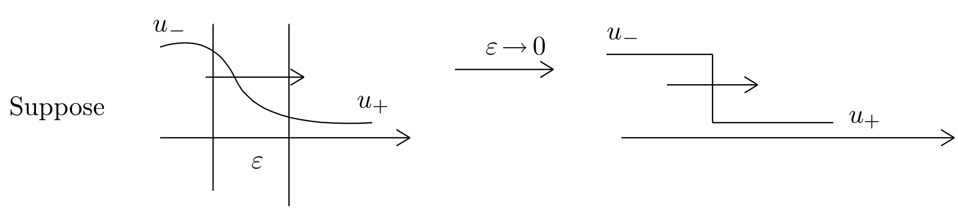

Figure 1.12.

Traveling wave solution is of the form

$$u^\varepsilon(x,t) = v\left(\frac{x - c\,t}{\varepsilon}\right),$$

where $c = [\![f(u)]\!]/[\![u]\!] = (u_- + u_+)/2$. And

$$-c\,v' + \left(\frac{v^2}{2}\right)' = v''.$$

Integrate and obtain

$$-c(v - u_-) + \frac{v^2}{2} - \frac{u_-^2}{2} = v'.$$

For a traveling wave

$$\begin{aligned} 2\varepsilon \int_{\mathbb{R}} (u_x^\varepsilon)^2 \mathrm{d}x &= 2\frac{\varepsilon}{\varepsilon} \int_{\mathbb{R}} \left(v'\left(\frac{x - c\,t}{\varepsilon}\right)\right)^2 \frac{\mathrm{d}x}{\varepsilon} \\ &= 2\int_{\mathbb{R}} (v')^2 \mathrm{d}x \end{aligned}$$

independent of ε! In fact,

$$\begin{aligned} 2\int_{\mathbb{R}} (v')^2 \mathrm{d}x &= 2\int_{\mathbb{R}} v' \cdot \frac{\mathrm{d}v}{\mathrm{d}x} \mathrm{d}x \\ &= 2\int_{u_-}^{u_+} \left[-c(v - u_-) + \left(\frac{v^2}{2} - \frac{u_-^2}{2}\right)\right] \mathrm{d}v \\ &\overset{(*)}{=} 2(u_- - u_+)^3 \int_0^1 s(1-s)\mathrm{d}s = \frac{(u_- - u_+)^3}{6}, \end{aligned}$$

where the step marked $(*)$ uses the Rankine-Hugoniot condition. We always have $u_- > u_+$. Heuristic picture:

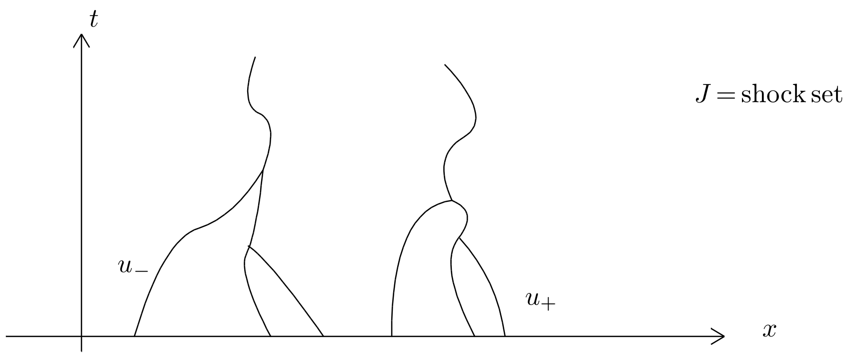

Figure 1.13.

The dissipation measure is concentrated on J and has density

$$\frac{(u_+ - u_-)^2}{6}.$$

1.5 Kružkov's uniqueness theorem

In what follows, $Q = \mathbb{R}^n \times (0, \infty)$. Consider entropy solutions to

$$\begin{aligned} u_t + D_x \cdot (f(u)) &= 0 \quad (x,t) \in Q \\ u(x,0) &= u_0(x) \end{aligned}$$

Here, $u: Q \to \mathbb{R}$, $f: \mathbb{R} \to \mathbb{R}$, $M := \|u\|_{L^\infty(Q)}$. Characteristics:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f'(u) \quad \text{or} \quad \frac{\mathrm{d}x_i}{\mathrm{d}t} = f_i(u), \quad i = 1, \ldots, n.$$

Let $c_* = \sup_{u \in [-M,M]} |f'(u)|$ be the maximum speed of characteristics. Consider the area given by

$$K_R = \left\{ (x,t) \colon |x| \leqslant R - c_* t, 0 \leqslant t \leqslant \frac{R}{c_*} \right\}$$

Define $r := R/c_*$.

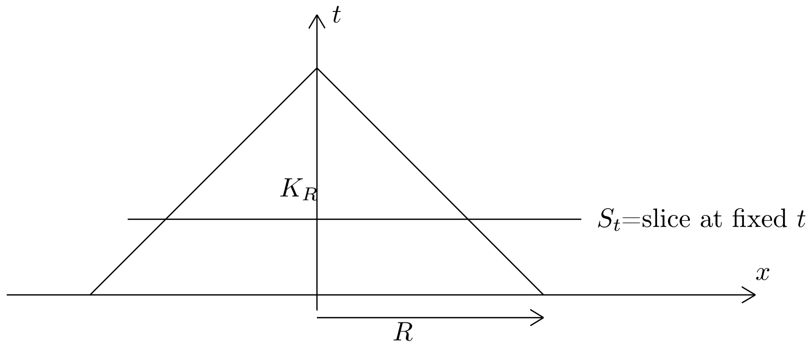

Figure 1.14.

Theorem 1.18. (Kružkov, 1970) *Suppose u, v are entropy solutions to the system such that*

$$\|u\|_{L^\infty(Q)}, \|v\|_{L^\infty(Q)} \leqslant M.$$

Then for almost every $t_1 < t_2$, $t_i \in [0,T]$, we have

$$\int_{S_{t_2}} |u(x,t_2) - v(x,t_2)| \mathrm{d}x \leqslant \int_{S_{t_1}} |u(x,t_1) - v(x,t_1)| \mathrm{d}x.$$

In particular, for a.e. $t \in [0,T]$

$$\int_{S_t} |u(x,t) - v(x,t)| \leqslant \int_{S_0} |u_0(x) - v_0(x)| \mathrm{d}x.$$

Corollary 1.19. *If $u_0 = v_0$, then $u = v$. (I.e. entropy solutions are unique, if they exist.)*

Proof. Two main ideas:

- doubling trick,
- clever choice of test functions.

Recall that if u is an entropy solution for every $\varphi \geqslant 0$ in $C_0^\infty(Q)$ and every $k \in \mathbb{R}$, we have

$$\int_Q [|u(x,t) - k|\varphi_t + \operatorname{sgn}(u-k)(f(u) - f(k)) \cdot D_x\varphi] \mathrm{d}x\, \mathrm{d}t \geqslant 0$$

Fix y, τ such that $v(y,\tau)$ is defined, let $k = v(y,\tau)$.

$$\int_Q [|u(x,t) - v(y,\tau)|\varphi_t + \operatorname{sgn}(u-v)(f(u) - f(v)) \cdot D_x\varphi] \mathrm{d}x\, \mathrm{d}t \geqslant 0.$$

This holds for (y,τ) a.e., so we have

$$\int_Q \int_Q [\text{as above}] \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}y\, \mathrm{d}\tau \geqslant 0.$$

Moreover, this holds for every $\varphi \in C_c^\infty(Q \times Q)$, with $\varphi \geqslant 0$. We also have a symmetric inequality with φ_τ, $D_y\varphi$ instead of φ_t, $D_x\varphi$. Add these to obtain

$$\int_Q \int_Q [|u(x,t) - v(y,\tau)|(\varphi_t + \varphi_\tau) + \operatorname{sgn}(u-v)(f(u) - f(v)) \cdot (D_x\varphi + D_y\varphi)] \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}y\, \mathrm{d}\tau \geqslant 0.$$

This is what is called the *doubling trick*. Fix $\psi \subset C_c^\infty(Q)$ and a "bump" function $\eta\colon \mathbb{R} \to \mathbb{R}$ with $\eta \geqslant 0$, $\int_{\mathbb{R}} \eta \mathrm{d}r = 1$. For $h > 0$, let $\eta_h(r) := 1/h\, \eta(r/h)$. Let

$$\psi(x,t,y,\tau) = \psi\left(\frac{x+y}{2}, \frac{t+\tau}{2}\right)\lambda_h\left(\frac{x-y}{2}, \frac{t-\tau}{2}\right)$$

where

$$\underbrace{\lambda_h(z,s)}_{\text{Approximate identity in } \mathbb{R}^n} = \eta_h(s) \prod_{i=1}^n \eta_h(z_i).$$

$$\begin{aligned} \varphi_t &= \frac{1}{2}\psi_t \cdot \lambda_h + \frac{1}{2}\psi(\lambda_h)_t \\ \varphi_\tau &= \frac{1}{2}\psi_t \lambda_h - \frac{1}{2}\psi(\lambda_h)_t \end{aligned}$$

Adding the two cancels out the last term:

$$\varphi_t + \varphi_\tau = \lambda_h \psi_t.$$

Similarly,

$$D_x\varphi + D_y\varphi = \lambda_h D_x\psi.$$

We then have

$$\int_Q \int_Q \lambda_h\left(\frac{x-y}{2}, \frac{t-\tau}{2}\right)\left[|u(x,t) - v(y,\tau)|\psi_t\left(\frac{x+y}{2}, \frac{t+\tau}{2}\right) + \operatorname{sgn}(u-v)(f(u) - f(v))D_x\psi\right]\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}\tau \geqslant 0$$

λ_h concentrates at $x = y$, $t = \tau$ as $h \to 0$.

Technical step 1. Let $h \to 0$. (partly outlined in homework, Problems 6 & 7)

$$\int_Q [|u(x,t) - v(x,t)|\psi_t + \operatorname{sgn}(u-v)(f(u) - f(v)) \cdot D_x\psi]\mathrm{d}x\,\mathrm{d}t \geqslant 0 \qquad (1.5)$$

[To prove this step, use Lebesgue's Differentiation Theorem.]

Claim: (1.5) $\Rightarrow L^1$ stability estimate. Pick two test functions:

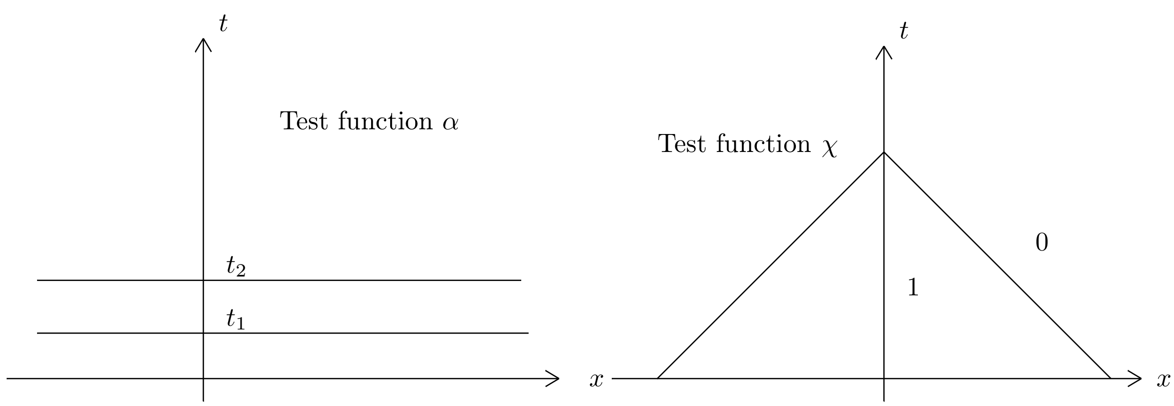

Figure 1.15.

Let

$$\alpha_h(t) = \int_{-\infty}^t \eta_h(r)\mathrm{d}r.$$

Choose

$$\psi(x,t) = (\alpha_h(t - t_1) - \alpha_h(t - t_2))\chi_\varepsilon(x,t).$$

where

$$\chi_\varepsilon = 1 - \alpha_\varepsilon(|x| + c_* t - R + \varepsilon).$$

Observe that

$$(\chi_\varepsilon)_t = -\alpha_\varepsilon'(c_*) \leqslant 0, \quad D_x\chi_\varepsilon = -\alpha_\varepsilon' \cdot \frac{x}{|x|}.$$

Therefore

$$(\chi_\varepsilon)_t + c_*|D_x\chi_\varepsilon| = -\alpha_\varepsilon' c_* + \alpha_\varepsilon' c_* = 0.$$

Drop ε:

$$\begin{aligned} & |u-v|\chi_t + \operatorname{sgn}(u-v)(f(u)-f(v)) \cdot D_x\chi \\ = \; & |u-v|\left[\chi_t + \frac{f(u)-f(v)}{u-v} \cdot D_x\chi\right] \leqslant |u-v|[\chi_t + c_*|D_x\chi|] = 0 \quad (\#\#) \end{aligned}$$

Substitute for ψ and use (##) to find

$$\int_Q (\alpha_h'(t-t_1) - \alpha_h'(t-t_2))|u-v|\chi \, \mathrm{d}x \, \mathrm{d}t \geqslant 0$$

$\Rightarrow L^1$ contraction. □

2 Hamilton-Jacobi Equations

$$u_t + H(x, Du) = 0$$

for $x \in \mathbb{R}^n$ and $t > 0$, with $u(x,0) = u_0(x)$. Typical application: Curve/surface evolution. (Think fire front.)

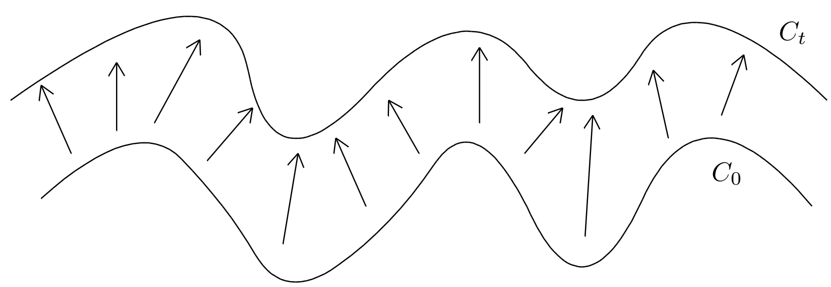

Figure 2.1.

Example 2.1. *(A curve that evolves with unit normal velocity)* If C_t is given as a graph $u(x,t)$. If τ is a tangential vector, then

$$\tau = \frac{(1, u_x)}{\sqrt{1+u_x^2}}.$$

Let $\dot{y} = u_t(x,t)$. So the normal velocity is

$$v_n = (0, \dot{y}) \cdot \nu,$$

where ν is the normal.

$$\nu = \frac{(u_x, -1)}{\sqrt{1+u_x^2}}.$$

Then $v_n = 1 \Rightarrow \dot{y}/\sqrt{1+u_x^2} = -1 \Rightarrow u_t = -\sqrt{1+u_x^2}$.

$$u_t + \sqrt{1+u_x^2} = 0$$

H is the *Hamiltonian*, which in this case is $\sqrt{1+u_x^2}$. In $\mathbb{R}^n$

$$u_t + \sqrt{1+|D_xu|^2} = 0,$$

a graph in $\mathbb{R}^n$.

Other rules for normal velocity can lead to equations with very different character.

Example 2.2. *(Motion by mean curvature)* Here $v_n = -\kappa$ (mean curvature).

$$\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}$$

$v_n = -\kappa$. Then

$$\frac{-u_t}{\sqrt{1+u_x^2}} = \frac{-u_{xx}}{(1+u_x)^{3/2}}.$$

So the equation is

$$u_t = \frac{u_{xx}}{(1+u_x^2)},$$

which is parabolic. Heuristics:

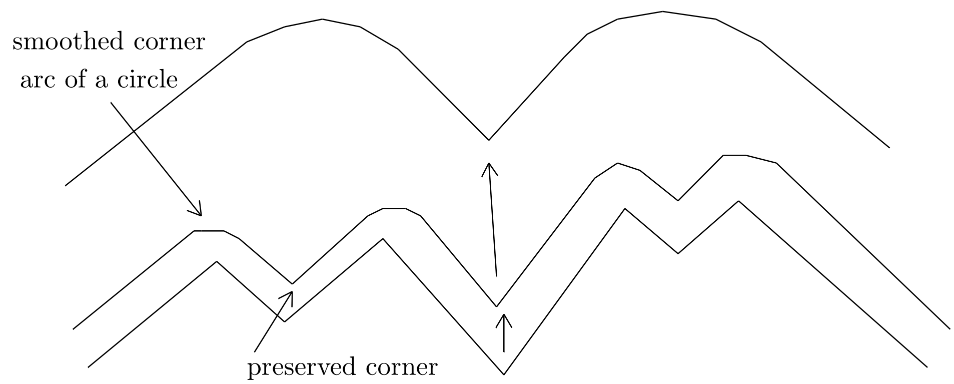

Figure 2.2.

If $(x,y) \in C_t$, then $\mathrm{dist}((x,y),C_0) = t$. Also

$$\partial_t u + \partial_x\left(\frac{u^2}{2}\right) = 0 \quad \xrightarrow{\text{integrate}} \quad U_t + \frac{U_x^2}{2} = 0.$$

2.1 Other motivation: Classical mechanics/optics

cf. Evans, chapter 3.3

- Newton's second law — $F = m\,a$
- Lagrange's equations
- Hamilton's equations

Lagrange's equations: State of the system $x \in \mathbb{R}^n$ or $\mathcal{M}^n$ (which is the configuration space). Then

$$L(x,\dot{x},t) = \underbrace{T}_{\text{kinetic}} - \underbrace{U(x)}_{\text{potential}}.$$

Typically, $T = \frac{1}{2} x \cdot M x$, where M is the (pos.def.) mass matrix.

Hamilton's principle: A path in configuration space between fixed states $x(t_0)$ and $x(t_1)$ *minimizes* the action

$$S(\Gamma) = \int_{t_0}^{t_1} L(x,\dot{x},t)\mathrm{d}t$$

over all paths $x(t) = \Gamma$.

Theorem 2.3. *Assume L is C^2. Fix $x(t_0)$, $x(t_1)$. If Γ is an extremum of S then*

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial x} = 0.$$

Proof. ("Proof") Assume that there is an optimal path $x(t)$. Then consider a perturbed path that respects the endpoints:

$$x_\varepsilon(t) = x(t) + \varepsilon\varphi(t)$$

with $\varphi(t_0) = \varphi(t_1) = 0$. Sicne $x(t)$ is an extremem of action,

$$\frac{\mathrm{d}S}{\mathrm{d}\varepsilon}(x(t) + \varepsilon\varphi(t))|_{\varepsilon=0} = 0.$$

So

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_{t_0}^{t_1} L(x + \varepsilon\varphi, \dot{x} + \varepsilon\dot{\varphi}, t)\mathrm{d}t,$$

which results in

$$\begin{aligned}
\int_{t_0}^{t_1}\left[\frac{\partial L}{\partial x}(x, \dot{x}, t)\varphi + \frac{\partial L}{\partial \dot{x}}(x, \dot{x}, t)\dot{\varphi}\right]\mathrm{d}t &= 0\\
\Rightarrow \int_{t_0}^{t_1}\varphi(t)\left[\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)\right]\mathrm{d}t + \underbrace{\frac{\partial L}{\partial \dot{x}}\varphi|_{t_0}^{t_1}}_{=0} &= 0
\end{aligned}$$

Since φ was arbitrary,

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial x} = 0. \qquad \square$$

Typical example: N-body problem

$$x = (y_1, \ldots, y_N), \quad y_i \in \mathbb{R}^3.$$

Then

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i |y_i|^2$$

and $U(x) =$ given potential, $L = T - U$. So

$$m_i \ddot{y}_{i,j} = -\frac{\partial U}{\partial y_{i,j}} \quad i = 1, \ldots, N, \quad j = 1, \ldots, 3.$$

2.1.1 Hamilton's formulation

$$H(x, p, t) = \underbrace{\sup_{y\in\mathbb{R}^n}(p\,y - L(x, y, t))}_{\text{Legendre transform}}$$

Then

$$\begin{aligned}
\dot{x} &= \frac{\partial H}{\partial p},\\
\dot{p} &= -\frac{\partial H}{\partial x},
\end{aligned}$$

called *Hamilton's equations*. They end up being $2N$ first-order equations.

Definition 2.4. *Suppose* $f: \mathbb{R}^n \to \mathbb{R}$ *is convex. Then the* Legendre transform *is*

$$\begin{aligned}
f^*(p) &:= \sup_{x\in\mathbb{R}^n}(p\cdot x - f(x))\\
&= \max_{x\in\mathbb{R}^n}(\ldots) \quad \textit{if} \quad \frac{f(x)}{|x|} \to \infty \quad \textit{as} \quad |x| \to \infty.
\end{aligned}$$

Example 2.5. $f(x) = \frac{1}{2}m\,x^2$, $m > 0$ and $x \in \mathbb{R}$.

$$(p\,x - f(x))' = 0 \Rightarrow (p - m\,x) = 0 \Rightarrow x = \frac{p}{m}.$$

And

$$f^*(p) = p\cdot\frac{p}{m} - \frac{1}{2}m\left(\frac{p}{m}\right)^2 = \frac{1}{2}\frac{p^2}{m}.$$

Example 2.6. $f(x) = \frac{1}{2} x \cdot M x$, where M is pos.def. Then

$$f^*(p) = \frac{1}{2} p \cdot M^{-1} p.$$

Example 2.7. Suppose $f(x) = x^\alpha / \alpha$ with $1 < \alpha < \infty$.

$$f^*(p) = \frac{p^\beta}{\beta}, \quad \text{where } \frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

Young's inequality and

$$f^*(p) + f(x) \geqslant p\, x$$

imply

$$\frac{x^\alpha}{\alpha} + \frac{p^\beta}{\beta} \geqslant p\, x.$$

Example 2.8.

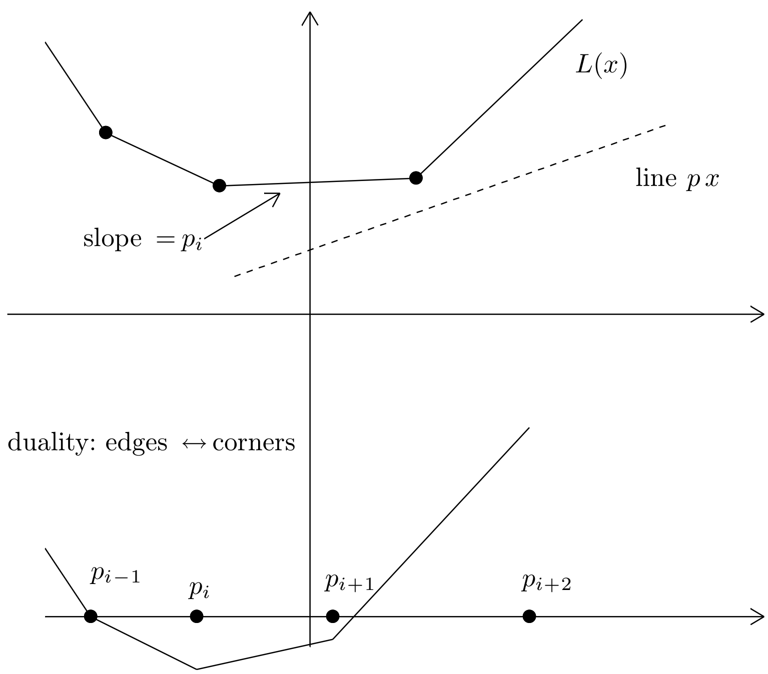

Figure 2.3.

Theorem 2.9. *Assume L is convex. Then $L^{**} = L$.*

Proof. see Evans. Sketch:

- If L_k is piecewise affine, then $L_k^{**} = L_k$ can be verified explicitly.
- Approximation: If $L_k \to L$ locally uniformly, then $L_k^* \to L^*$ locally uniformly. $\square$

Back to Hamilton-Jacobi equations:

$$u_t + H(x, D_x u, t) = 0.$$

H is always assumed to be

- $C^2(\mathbb{R}^n \times \mathbb{R}^n \times [0, \infty))$,
- uniformly convex in $p = D_x u$,
- uniformly superlinear in p.

2.1.2 Motivation for Hamilton-Jacobi from classical mechanics

Principle of least action: For every path connecting $(x_0, t_0) \to (x_1, t_1)$ associate the 'action'

$$S(\Gamma) = \int_{\Gamma} L(x, \dot{x}, t)\mathrm{d}t.$$

L Lagrangian, convex, superlinear in $\dot{x}$. Least action $\Rightarrow$ Lagrange's equations:

$$-\frac{\mathrm{d}}{\mathrm{d}t}[D_{\dot{x}}L(x, \dot{x}, t)] + D_x L = 0 \tag{2.1}$$

$x \in \mathbb{R}^n \Rightarrow n$ 2nd order ODE.

Theorem 2.10. *("Theorem") (2.1) is equivalent to*

$$\dot{x} = D_p H, \quad \dot{p} = -D_x H. \tag{2.2}$$

Note that those are $2n$ first order ODEs.

Proof. ("Proof")

$$H(x, p, t) = \max_{v \in \mathbb{R}^n} (v p - L(x, v, t)).$$

Maximum is attained when

$$p = D_v L(x, v, t), \tag{2.3}$$

and the solution is unique because of convexity.

$$H(x, p, t) = v(x, p, t)p - L(x, v(x, p, t), t),$$

where v solves (2.3).

$$\begin{aligned} D_p H &= v + p\, D_p v - D_v L \cdot D_p v \\ &= v + \underbrace{(p - D_v L)}_{=0 \text{ because of (2.3)}} D_p v \\ &= v. \end{aligned}$$

Thus $\dot{x} = D_p H$. Similarly, we use (2.1)

$$\frac{\mathrm{d}}{\mathrm{d}t}(p) = D_x L$$

Note that

$$\begin{aligned} D_x H &= p\, D_x v - D_x L - D_v L\, D_x v \\ &= -D_x L + \underbrace{[p - D_v L]}_{=0 \text{ because of (2.3)}} D_x v. \end{aligned}$$

Thus, $\dot{p} = -D_x H$. □

Connections to Hamilton-Jacobi:

- (2.2) are characteristics of Hamilton-Jacobi equations.
- If $u = S(\Gamma)$, then $\mathrm{d}u = p\ \mathrm{d}x - H\ \mathrm{d}t$. (cf. Arnold, "Mathematical Methods in Classical Mechanics", Chapter 46)

$$\left\{ \frac{\partial u}{\partial t} = -H(x, p, t); \quad D_x u = p \right\} \quad \Rightarrow \quad u_t + H(x, Du, t) = 0.$$

Important special case: $H(x, p, t) = H(p)$.

Example 2.11. $u_t - \sqrt{1 + |D_x u|^2 = 0}$. $H(p) = -\sqrt{1 + |p|^2}$.

Example 2.12. $u_t + \frac{1}{2}|D_x u|^2 = 0$. $H(p) = \frac{1}{2}|p|^2$.

$$\left\{ \begin{array}{l} \dot{x} = D_p H(p) \\ \dot{p} = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} p(t) = p(0) \\ x(t) = x(0) + D_p H(p(0)) \end{array} \right. \quad \rightarrow \quad \text{straight line characteristics!}$$

2.2 The Hopf-Lax Formula

$$u_t + H(D_x u) = 0, \quad u(x,0) = u_0(x) \tag{2.4}$$

for $x \in \mathbb{R}^n$, $t > 0$. Always, H is considered convex and superlinear, $L = H^*$. Action on a path connecting $x(t_0) = y$ and $x(t_1) = x$:

$$\int_{t_0}^{t_1} L(x, \dot{x}, t)\mathrm{d}t = \int_{t_0}^{t_1} L(\dot{x}(t))\mathrm{d}t \geqslant (t_1 - t_0) L\left(\frac{x-y}{t_1 - t_0}\right).$$

Using Jensen's inequality:

$$\frac{1}{t_1 - t_0}\int_{t_0}^{t_1} L(\dot{x})\mathrm{d}t \geqslant L\left(\frac{1}{t_1 - t_0}\int_{t_0}^{t_1} \dot{x}\,\mathrm{d}t\right) = L\left(\frac{x(t_1) - x(t_0)}{t_1 - t_0}\right).$$

Hopf-Lax formula:

$$u(x,t) = \min_{y \in \mathbb{R}^n}\left[t\,L\left(\frac{x-y}{t}\right) + u_0(y)\right]. \tag{2.5}$$

Theorem 2.13. *Assume $u_0: \mathbb{R}^n \to \mathbb{R}$ is Lipschitz with $\mathrm{Lip}(u(\,\cdot\,, t)) \leqslant M$ Then u defined by (2.5) is Lipschitz in $\mathbb{R}^n \times [0, \infty)$ and solves (2.4) a.e.. In particular, u solves (2.4) in $\mathcal{D}'$.*

(Proof exacty follows Evans.)

Lemma 2.14. (Semigroup Property)

$$u(x,t) = \min_{y \in \mathbb{R}^n}\left[(t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s)\right]$$

where $0 \leqslant s < t$.

Proof.

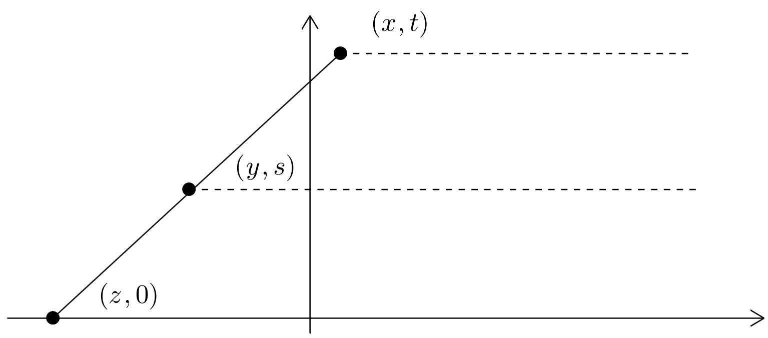

Figure 2.4.

$$\frac{x-z}{t} = \frac{x-y}{t-s} = \frac{y-z}{s}$$

So

$$\frac{x-z}{t} = \left(1 - \frac{s}{t}\right)\left(\frac{x-y}{t-s}\right) + \frac{s}{t}\left(\frac{y-z}{s}\right).$$

Since L is convex,

$$L\left(\frac{x-z}{t}\right) \leqslant \left(1 - \frac{s}{t}\right) L\left(\frac{x-y}{t-s}\right) + \frac{s}{t} L\left(\frac{y-z}{t}\right).$$

Choose z such that

$$u(y,s) = s\,L\left(\frac{y-z}{t}\right) + u_0(z).$$

The minimum is achieved because L is superlinear. Also,

$$\frac{|u_0(y) - u_0(0)|}{|y|} \leqslant M$$

because u_0 is Lipschitz.

$$t\,L\left(\frac{x-z}{t}\right)+u_0(z)\leqslant(t-s)L\left(\frac{x-y}{t-s}\right)+u(y,s).$$

But

$$u(x,t)=\min_{z'}\left[t\,L\left(\frac{x-z'}{t}\right)+u_0(z')\right].$$

Thus

$$u(x,t)\leqslant(t-s)L\left(\frac{x-y}{t-s}\right)+u(y-s)$$

for all $y\in\mathbb{R}^n$. So,

$$u(x,t)\leqslant\min_{y\in\mathbb{R}^n}\left[(t-s)L\left(\frac{x-y}{t-s}\right)+u(y-s)\right].$$

To obtain the opposite inequality, choose z such that

$$u(x,t)=t\,L\left(\frac{x-z}{t}\right)+u_0(z).$$

Let $y=(1-s/t)z+(s/t)x$. Then

$$\begin{aligned}
u(y,s)+(t-s)\,L\left(\frac{x-y}{t-s}\right) &= u(y,s)+(t-s)L\left(\frac{x-z}{t}\right)\\
&= u(y,s)-s\,L\left(\frac{y-z}{s}\right)+[u(x,t)-u_0(z)]\\
&= u(y,s)-\left(u_0(z)+s\,L\left(\frac{y-z}{s}\right)\right)+u(x,t)\\
&\leqslant u(x,t).
\end{aligned}$$

That means

$$\min_{y\in\mathbb{R}^n}\left[(t-s)L\left(\frac{x-y}{t-s}\right)+u(y-s)\right]\leqslant u(x,t).$$

□

Lemma 2.15. *$u:\mathbb{R}^n\times[0,\infty)\to\mathbb{R}$ is uniformly Lipschitz. On any slice $t=\mathrm{const}$ we have*

$$\mathrm{Lip}(u(\,\cdot\,,t))\leqslant M.$$

Proof. (1) Fix $x,\hat{x}\in\mathbb{R}^n$. Choose $y\in\mathbb{R}^n$ such that

$$\begin{aligned}
u(x,t) &= t\,L\left(\frac{x-y}{t}\right)+u_0(y),\\
u(\hat{x},t) &= t\,L\left(\frac{\hat{x}-y}{t}\right)+u_0(y).
\end{aligned}$$

Then

$$u(\hat{x},t)-u(x,t)=\inf_{z\in\mathbb{R}^n}\left[t\,L\left(\frac{\hat{x}-z}{t}\right)+u_0(z)\right]-\left[t\,L\left(\frac{x-y}{t}\right)+u_0(y)\right].$$

Choose z such that

$$\begin{aligned}
\hat{x}-z &= x-y\\
\Leftrightarrow z &= \hat{x}-x+y.
\end{aligned}$$

Then

$$\begin{aligned}
u(\hat{x},t)-u(x,t) &\leqslant u_0(\hat{x}-x+y)-u_0(y)\\
&\leqslant M|\hat{x}-x|,
\end{aligned}$$

where $M=\mathrm{Lip}(u_0)$. Similarly,

$$u(x,t)-u(\hat{x},t)\leqslant M|x-\hat{x}|.$$

This yields the Lipschitz claim. In fact, using Lemma 2.14 we have

$$\mathrm{Lip}(u(\,\cdot\,,t)) \leqslant \mathrm{Lip}(u(\,\cdot\,,s))$$

for every $0 \leqslant s < t$, which can be seen as "the solution is getting smoother".

(2) *Smoothness in t:*

$$u(x,t) = \min_y \left[t\,L\!\left(\frac{x-y}{t}\right) + u_0(y)\right] \leqslant t\,L(0) + u_0(x) \quad (\text{choose } y = x). \tag{2.6}$$

Then

$$\frac{u(x,t) - u_0(x)}{t} \leqslant L(0).$$

$$|u_0(y) - u_0(x)| \leqslant M\,|x-y| \quad \Rightarrow \quad u_0(y) \geqslant u_0(x) - M\,|x-y|.$$

Thus

$$t\,L\!\left(\frac{x-y}{t}\right) + u_0(y) \geqslant t\,L\!\left(\frac{x-y}{t}\right) + u_0(x) - M\,|x-y|.$$

By (2.6),

$$\begin{aligned}
u(x,t) - u_0(x) &\geqslant \min_y \left[t\,L\!\left(\frac{x-y}{t}\right) - M\,|x-y|\right]\\
&= -t \max_{z\in\mathbb{R}^n} [M\,|z| - L(z)]\\
&= -t \max_{z\in\mathbb{R}^n} \left[\max_{\omega\in B(0,M)} \omega\cdot z - L(z)\right]\\
&= -t \max_{\omega\in B(0,M)} \max_{z\in\mathbb{R}^n} [\omega\cdot z - L(z)]\\
&= -t \max_{\omega\in B(0,M)} H(\omega).
\end{aligned}$$

Now

$$-\max_{\omega\in B(0,M)} H(\omega) \leqslant \frac{u(x,t) - u_0(x)}{t} \leqslant L(0),$$

where both the left and right term only depend on the equation. $\Rightarrow$ Lipschitz const in time $\leqslant \max\,(L(0), \max_{\omega\in B(0,M)} H(\omega))$. □

(Feb 22) Let $Q := \mathbb{R}^n \times (0,\infty)$.

Theorem 2.16. *u satisfies (2.4) almost everywhere in Q.*

Proof. 1) We will use Rademacher's Theorem, which says $u \in \mathrm{Lip}(Q) \Rightarrow u$ is differentiable a.e. (i.e., in Sobolev space notation, $W^{1,\infty}(Q) = \mathrm{Lip}(Q)$.)

2) We'll assume Rademacher's Theorem and show that (2.4) holds at any $(x,\,t)$ where u is differentiable. Fix (x,t) as above. Fix $q\in\mathbb{R}^n$, $h>0$. Then

$$u(x+h\,q, t+h) \overset{(\text{Lemma 2.14})}{=} \min_y \left[h\,L\!\left(\frac{x+h\,q-y}{h}\right) + u(y,t)\right].$$

Choose $y = x$. Then

$$u(x+h\,q, t+h) \;\leqslant\; h\,L(q) + u(x,t)$$

and

$$\frac{u(x+h\,q, t+h)}{h} + \frac{u(x,t+h) - u(x,t)}{h} \leqslant L(q).$$

So, if we let $h \searrow 0$, we have $D_x u\cdot q + u_t \leqslant L(q)$. Then

$$u_t \leqslant -[D_x u\cdot q - L(q)],$$

since q is arbitrary, optimize bound to become

$$u_t \leqslant -H(D_x u).$$

[Quick reminder: We want

$$u_t = -H(D_x u).$$

We already have one side of this.] Now for the converse inequality: Choose z such that

$$u(x,t) = L\left(\frac{x-z}{t}\right) + u_0(z).$$

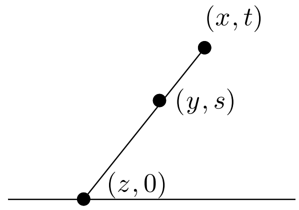

Figure 2.5.

Fix $h > 0$, let $s = t - h$. Then

$$y = \left(1 - \frac{s}{t}\right) z + \frac{s}{t} x = \frac{h}{t} z + \left(1 - \frac{h}{t}\right) x$$

and observe

$$\begin{aligned} u(y,s) &= \min_{z'} \left[s\,L\left(\frac{y-z'}{s}\right) + u_0(z') \right] \leqslant s\,L\left(\frac{y-z}{s}\right) + u_0(z) \\ \Rightarrow -u(y,s) &\geqslant -\left[s\,L\left(\frac{y-z}{s}\right) + u_0(z) \right]. \end{aligned}$$

to find

$$\begin{aligned} u(x,t) - u(y,s) &\geqslant t\,L\left(\frac{x-z}{t}\right) + \cancel{u_0(z)} - \left[s\,L\left(\frac{y-z}{t}\right) + \cancel{u_0(z)} \right] \\ \Rightarrow u(x,t) - u(y,s) &\geqslant h\,L\left(\frac{x-z}{t}\right) \\ \Rightarrow \frac{u(x,t) - u\left(x - \frac{h}{t}(x-z), t-h\right)}{h} &\geqslant L\left(\frac{x-z}{t}\right). \end{aligned}$$

Let $h \searrow 0$. Then

$$\begin{aligned} u_t + D_x u\left(\frac{x-z}{t}\right) &\geqslant L\left(\frac{x-z}{t}\right) \\ u_t &\geqslant L\left(\frac{x-z}{t}\right) - D_x u \cdot \left(\frac{x-z}{t}\right) \geqslant -H(D_x u). \end{aligned}$$

□

2.3 Regularity of Solutions

Consider again surface evolution: $u_t - \sqrt{1 + |D_x u|^2} = 0$ (note the concave Hamiltonian). The surface evolves with unit normal velocity. So far, $\mathrm{Lip}(u(\,\cdot\,,t)) \leqslant \mathrm{Lip}(u(\,\cdot\,,s))$ for any $s \leqslant t$.

"One sided second derivative":

Definition 2.17. (Semiconcavity) $f\colon \mathbb{R}^n \to \mathbb{R}$ *is semiconcave if* $\exists c > 0$

$$f(x+z) - 2f(x) + f(x-z) \leqslant C|z|^2$$

for every $x, z \in \mathbb{R}^n$.

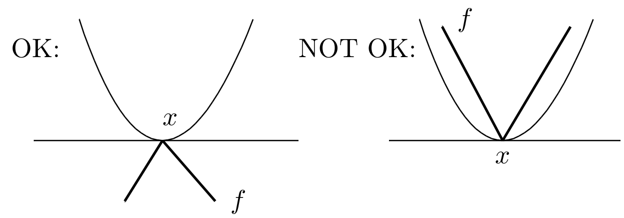

Figure 2.6. Semiconcavity.

In the example, u is semiconvex (because $H(p) = -\sqrt{1+|p|^2}$, so signs change).

Definition 2.18. *H is* uniformly convex *if there is a constant $\theta > 0$ such that*

$$\xi^t D^2 H(p) \xi \geqslant \theta |\xi|^2$$

for every $p, \xi \in \mathbb{R}^n$.

Theorem 2.19. *Assume H is uniformly convex. Then*

$$u(x+z,t) - 2u(x,t) + u(x-z,t) \leqslant \frac{1}{\theta t}|z|^2 \quad (\forall x \in \mathbb{R}^n, t > 0).$$

Proof. 1) Because H is uniformly convex, we have

$$H\left(\frac{p_1+p_2}{2}\right) \leqslant \underbrace{\frac{1}{2}H(p_1) + \frac{1}{2}H(p_2)}_{\text{from convexity}} + \underbrace{\frac{\theta}{8}|p_1-p_2|^2}_{\text{from } uniform \text{ convexity}}.$$

So,

$$\frac{1}{2}(L(q_1) + L(q_2)) \leqslant L\left(\frac{q_1+q_2}{2}\right) + \frac{1}{8\theta}|q_1-q_2|^2. \tag{2.7}$$

To see this, choose p_i such that $H(p_i) = p_i q_i - L(q_i)$. Then

$$\frac{1}{2}(H(p_1) + H(p_2)) = \frac{1}{2}(p_1 q_1 + p_2 q_2) - \frac{1}{2}(L(q_1) + L(q_2)).$$

This yields (2.7).

2) Choose y such that

$$u(x,t) = t\,L\left(\frac{x-y}{t}\right) + u_0(y).$$

By the Hopf-Lax formula,

$$\begin{aligned} u(x+z,t) - 2u(x,t) + u(x-z,t) &\leqslant t\,L\left(\frac{x+z-y}{t}\right) + 2u_0(y) - 2t\,L\left(\frac{x-y}{t}\right) - 2u_0(y) \\ &= 2t\left[\frac{1}{2}L\left(\frac{x+z-y}{t}\right) - L\left(\frac{x-y}{t}\right) + \frac{1}{2}L\left(\frac{x-y-z}{t}\right)\right] \\ &\overset{(2.7)}{\leqslant} 2t\frac{1}{8\theta}\left|\frac{2z}{t}\right|^2 = \frac{1}{\theta t}|z|^2 \end{aligned}$$

□

2.4 Viscosity Solutions

(cf. Chapter 10 in Evans) Again, let $Q := \mathbb{R}^n \times (0, \infty)$ and consider

$$u_t + H(D_x u, x) = 0, \quad u(x,0) = u_0(x). \tag{2.8}$$

Suppose

1. $H(p,x) \neq H(p)$,
2. There is no convexity on H.

Basic question: The weak solutions are non-unique. What is the 'right' weak solution?

Definition 2.20. (Crandall, Evans, P.L. Lions) $u \in \mathrm{BC}(\mathbb{R}^n \times [0, \infty)$ *is a* viscosity solution *provided*

1. $u(x, 0) = u_0(x)$
2. *For test functions* $v \in C^\infty(Q)$*:*
   A) *If* $u - v$ *has a local maximum at* (x_0, t_0)*, then* $v_t + H(D_x v, x) \leqslant 0$*,*
   B) *if* $u - v$ *has a local minimum at* (x_0, t_0)*, then* $v_t + H(D_x v, x) \geqslant 0$*.*

Remark 2.21. If u is a C^1 solution to (2.8), then it is a viscosity solution. Therefore suppose $u - v$ has a max at (x_0, t_0). Then

$$\begin{array}{ll} \partial_t(u - v) = 0 & D_x(u - v) = 0 \\ \partial_t u = \partial_t v & D_x u = D_x v \end{array} \quad \text{at } (x_0, t_0)$$

Since u solves (2.8), $v_t + H(D_x v, x)|_{(x_0, t_0)} = 0$ as desired.

Remark 2.22. The definition is unusual in the sense that 'there is no integration by parts' in the definition.

Theorem 2.23. (Crandall, Evans, Lions) *Assume there is* $C > 0$ *such that*

$$\begin{aligned} |H(x, p_1) - H(x, p_2)| &\leqslant C|p_1 - p_2| \\ |H(x_1, p) - H(x_2, p)| &\leqslant C(1 + |p|)|x_1 - x_2| \end{aligned}$$

for all $x \in \mathbb{R}^n$ *and* $p \in \mathbb{R}^n$*. If a vicosity solution exists, it is unique.*

Remark 2.24. Proving uniqueness is the hard part of the preceding theorem. Cf. Evans for complete proof. It uses the doubling trick of Kružkov.

What we will prove is the following:

Theorem 2.25. *If* u *is a viscosity solution, then* $u_t + H(D_x u, x) = 0$ *at all points where* u *is differentiable.*

Corollary 2.26. *If* u *is Lipschitz and a viscosity solution, then* $u_t + H(D_x u, x) = 0$ *almost everywhere.*

Proof. Lipschitz $\overset{\text{Rademacher}}{\Rightarrow}$ differentiable a.e. □

Lemma 2.27. (Touching by a C^1 function) *Suppose* $u: \mathbb{R}^n \rightarrow \mathbb{R}$ *is differentiable at* (x_0, t_0)*, then there is a* C^1 *function* $v: \mathbb{R}^n \rightarrow \mathbb{R}^n$ *such that* $u - v$ *has a strict maximum at* (x_0, t_0)*.*

Proof. (of Theorem 2.25) 1) Suppose u is differentiable at (x_0, t_0). Choose v touching u at (x_0, t_0) such that $u - v$ has a strict maximum at (x_0, t_0).

2) Pick a standard mollifier η, let η_ε be the L^1 rescaling. Let $v^\varepsilon = \eta_\varepsilon * v$. Then

$$\begin{cases} v^\varepsilon & \rightarrow & v \\ v_t^\varepsilon & \rightarrow & v_t \\ D_x v^\varepsilon & \rightarrow & D_x v \end{cases} \quad \text{uniformly on compacts as } \varepsilon \rightarrow 0.$$

Claim: $u - v^\varepsilon$ has a local maximum at some $(x_\varepsilon, t_\varepsilon)$ such that $(x_\varepsilon, t_\varepsilon) \rightarrow (x_0, t_0)$. (Important here: *strict* maximum assumption.)

Proof: For any r, there is a ball $B((x_0, t_0), r)$ such that $(u - v)(x_0, t_0) > \max_{\partial B} (u - v)$. So, for ε sufficiently small $(u - v^\varepsilon)(x_0, t_0) > \max_{\partial B} (u - v^\varepsilon)$. Then there exists some $(x_\varepsilon, t_\varepsilon)$ in the ball such that $u - v^\varepsilon$ has a local maximum. Moreover, letting $r \rightarrow 0$, we find $(x_\varepsilon, t_\varepsilon) \rightarrow (x_0, t_0)$.

(3) We use the definition of viscosity solutions to find

$$\begin{aligned} v_t^\varepsilon + H(D_x v^\varepsilon, x) &\leqslant 0 \quad \text{at } (x_\varepsilon, t_\varepsilon) \\ \Rightarrow v_t + H(D_x v, x) &\leqslant 0 \quad \text{at } (x_0, t_0). \end{aligned}$$

But $u - v$ is a local max $\Rightarrow D_x u = D_x v$, $u_t = v_t$. So,

$$u_t + H(D_x u, x) \leqslant 0.$$

(4) Similarly, use v touching from above to obtain the opposite inequality. $\square$

Digression: Why this definition?

- Semiconcavity
- Maximum principle (Evans)

If H were convex and $H(p)$, once again:

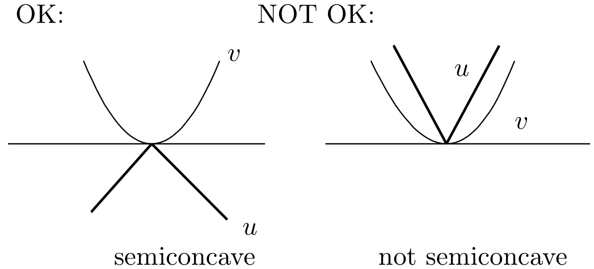

Figure 2.7. Semiconcavity

Proof. (of Lemma 2.27)

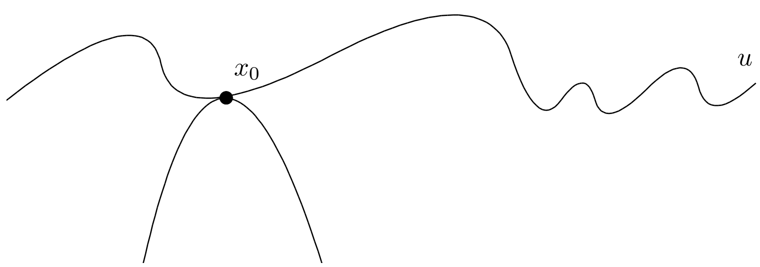

Figure 2.8.

We want $v \in C^1$ such that $u - v$ has a strict maximum at x_0. We know that u is differentiable at x_0 and continuous. Without loss, suppose $x_0 = 0$, $u(x_0) = 0$, $Du(x_0) = 0$. If not, consider

$$\tilde{u}(x) = u(x + x_0) - u(x_0) - Du(x_0)(x - x_0).$$

We can write $u(x) = |x|\rho_1(x)$, where $\rho_1(x)$ is continuous and $\rho_1(0) = 0$. Let

$$\rho_2(r) = \max_{|x| \leqslant r} |\rho_1(x)|.$$

$\rho_2: [0, \infty) \to [0, \infty)$ is continuous with $\rho_2(0) = 0$. Then set

$$v(x) = \int_{|x|}^{2|x|} \rho_2(r) \mathrm{d}r - |x|^2.$$

Clearly $v(0) = 0$,

$$v(0) = 0, Dv = \frac{2x}{x}\rho_2(2|x|) - \frac{x}{|x|}\rho_2(|x|) - 2x.$$

So, it is continuous and $Dv(0) = 0$. (just check) $\square$

3 Sobolev Spaces

Let $\Omega \subset \mathbb{R}^n$ be open. Also, let $D^\alpha u$ be the distributional derivative, with α a multi-index. $\partial^\alpha u$ shall be the classical derivative (if it exists).

Definition 3.1. *Let $k\in\mathbb{N}$ and $p\geqslant 1$. Let*

$$W^{k,p}(\Omega) := \{u\in\mathcal{D}' \colon D^\alpha u\in L^p(\Omega), |\alpha|\leqslant k\}.$$

If $u\in W^{k,p}(\Omega)$, we denote its norm by

$$\|u\|_{k,p;\Omega} := \sum_{|\alpha|\leqslant k} \|D^\alpha u\|_{L^p(\Omega)}.$$

Definition 3.2. *$W_0^{k,p}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in the $\|\cdot\|_{k,p;\Omega}$-norm.*

Proposition 3.3. *$W^{k,p}(\Omega)$ is a Banach space.*

Proposition 3.4. *Suppose $u\in W_0^{1,p}(\Omega)$. Define*

$$\tilde{u}(x) = \begin{cases} u(x) & x\in\Omega,\\ 0 & x\notin\Omega.\end{cases}$$

Then $\tilde{u}\in W_0^{1,p}(\mathbb{R}^n)$. (Extension by zero for $W_0^{1,p}(\Omega)$ is OK.)

Choose a standard mollifier $\psi\in C_c^\infty(\mathbb{R}^n)$ with $\psi\geqslant 0$, $\operatorname{supp}(\psi)\subset B(0,1)$, $\int_{\mathbb{R}^n}\psi\,\mathrm{d}x = 1$. For $\varepsilon>0$, let

$$\psi_\varepsilon(x) := \frac{1}{\varepsilon^n}\psi(x/\varepsilon).$$

Theorem 3.5. *Suppose $u\in W^{l,p}(\Omega)$. For every open $\Omega'\subset\subset\Omega$, there exist $u_k\in C_c^\infty(\Omega')$ such that*

$$\|u_k - u\|_{1,p;\Omega'}\to 0.$$

Proof. Let $\varepsilon_0 = \operatorname{dist}(\overline{\Omega'},\partial\Omega)$. Choose $\varepsilon_k\searrow 0$, with $\varepsilon_k<\varepsilon_0$. Set

$$u_k(x) = \psi_{\varepsilon_k} * u$$

for $x\in\Omega'$. We have $D^\alpha u_k = D^\alpha\psi_{\varepsilon_k} * u = \psi_{\varepsilon_k} * D^\alpha u$, for every α. Moreover, for $|\alpha|\leqslant l$, we have $D^\alpha u_k\to D^\alpha u$ in $L^p(\Omega')$. □

Typical idea in the theory: We want to find a representation of an equivalence class that has classical properties. *Example:* If $f\in L^1(\mathbb{R}^n)$, set

$$f_*(x) = \lim_{r\to 0}\frac{1}{B(x,r)}\int_{B(x,r)} f(y)\,\mathrm{d}y.$$

Theorem 3.6. *Suppose $u\in W^{1,p}(\Omega)$, $1\leqslant p\leqslant\infty$. Let $\Omega'\subset\subset\Omega$.*

1. *Then u has a representative u_* on Ω' that is absolutely continuous on a line parallel to the coordinate axes almost everywhere, and*

$$\partial_{x_i} u_* = D_{x_i} u \quad \text{a.e. for any } i = 1,\ldots,n.$$

2. *Conversely, if u has such a representative with $\partial^\alpha u^*\in L^p(\Omega')$, $|\alpha|\leqslant 1$, then $u\in W^{1,p}(\Omega)$.*

Why do we care? Two examples:

Corollary 3.7. *If Ω is connected, and $Du = 0$, then u is constant.*

Corollary 3.8. *Suppose $u,v\in W^{1,p}(\Omega)$. Then $\max\{u,v\}$ and $\min\{u,v\}$ are in $W^{1,p}(\Omega)$, and we have*

$$D\max\{u,v\} = \begin{cases} Du & \text{on } \{u\geqslant v\},\\ Dv & \text{on } \{u<v\}.\end{cases}$$

Proof. Choose representatives u_*, v_*. Then $\max\{u_*,v_*\}$ is absolutely continuous. □

Corollary 3.9. $u_+ = \max\{u, 0\} \in W^{1,p}(\Omega)$. *Likewise for* u_-.

Corollary 3.10. $u \in W^{1,p}(\Omega) \Rightarrow |u| \in W^{1,p}(\Omega)$.

Proof. $|u| = \max\{u_+, u_-\}$. □

Proof. (of Theorem 3.6) 1) Without loss of generality, suppose $\Omega = \mathbb{R}^n$, and u has compact support. We may as well set $p = 1$ because of Jensen's inequality. Pick $\chi \in C_c^\infty(\mathbb{R}^n)$ with $\chi = 1$ on Ω' and consider $\tilde{u} = \chi u$, and extend by 0.

2) Choose regularizations u_k such that

a) $\operatorname{supp}(u_k) \subset B(0, R)$ fixed,

b) $\|u_k - u\|_{1,p} < 2^{-k}$.

Set

$$G = \left\{x \in \mathbb{R}^n\colon \lim_{k\to\infty} u_k(x) \text{ exists}\right\}$$

and

$$u_*(x) = \lim_{k\to\infty} u(x)$$

for $x \in G$. We'll show that $|\mathbb{R}^n \setminus G| = 0$. Fix a coordinate direction, say $(0, \ldots, 0, 1)$. Write $x \in \mathbb{R}^n = (y, x_n)$ with $y \in \mathbb{R}^{n-1}$. Let

$$f_k(y) = \sum_{|\alpha| \leqslant 1} \int_{\mathbb{R}} |D^\alpha(u_{k+1} - u_k)|(y, x)\mathrm{d}x_n$$

Also let

$$f(y) = \sum_{k=1}^\infty f_k(y).$$

Observe that

$$\int_{\mathbb{R}^{n-1}} f(y)\,\mathrm{d}y \overset{\text{Fubini}}{=} \sum_{k=1}^\infty \int_{\mathbb{R}^n} \sum_{|\alpha| \leqslant 1} |D^\alpha(u_{k+1} - u_k)|\mathrm{d}x = \sum_{k=1}^\infty \|u_{k+1} - u_k\|_{1,1} \leqslant \sum_{k=1}^\infty \frac{1}{2^k} < \infty.$$

Then $f < \infty$ for $y \in \mathbb{R}^{n-1}$ a.e. Fix y s.t. $f(y) < \infty$. This implies

$$\lim_{k\to\infty} f_k(y) = 0.$$

Let $g_k(t) = u_k(y, t)$ for $t \in \mathbb{R}$. Then

$$g_k(t) - g_{k+1}(t) = \int_{-\infty}^t \partial_{x_n}(u_{k+1} - u_k)(y, x_n)\mathrm{d}x_n.$$

Thus

$$|g_k(t) - g_{k+1}(t)| \leqslant \int_{-\infty}^t |\partial_{x_n}(u_{k+1} - u_k)(y, x_n)|\mathrm{d}x_n \leqslant f_k(y)$$

uniformly in t. Thus

$$\lim_{k\to\infty} g_k(t) = \lim_{k\to\infty} u_k(y, t) = u_*(y, t)$$

is a continuous function of t. We may write

$$\begin{array}{rcl} g_k(t) & = & \displaystyle\int_{-\infty}^t \underbrace{g_k'(x_n)}\mathrm{d}x_n \\ \downarrow & & \qquad\;\downarrow \;\text{ (Cauchy sequence in } L^1(\mathbb{R})) \\ u_*(y, t) & = & \text{an } L^1 \text{ function } h. \end{array}$$

Thus

$$u_*(y, t) = \int_{-\infty}^t h(x_n)\mathrm{d}x_n$$

for every $t \in \mathbb{R}$. Thus u_* is absolutely continuous on the line $y = \text{const}$. □

Theorem 3.11. (Density of $C^\infty(\Omega)$) *Let* $1 \leqslant p < \infty$. *Let*

$$\mathcal{S}_p := \left\{ u : u \in C^\infty(\Omega), \|u\|_{1,p} < \infty \right\}.$$

Then $\overline{\mathcal{S}_p} = W^{1,p}(\Omega)$.

Remark 3.12. The above theorem is stronger than the previous approximation theorem 3.5, which was only concerned with compactly contained subsets $\Omega' \subset\subset \Omega$.

Proof. (Sketch, cf. Evans for details) Use partition of unity and previous approximation theorem. The idea is to exhaust Ω by $\bar{\Omega}_k \subset \Omega_{k+1}$ for which $\bigcup_{k=1}^\infty \Omega_k$, for example

$$\Omega_k := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > 1/k\}.$$

Choose partition of unity subordinate to

$$G_k = \Omega_k \setminus \bar{\Omega}_{k-1}, \quad \Omega_0 = \emptyset$$

and previous theorem on mollification. □

3.1 Campanato's Inequality

Theorem 3.13. (Campanato) *Suppose* $u \in L^1_{\text{loc}}(\Omega)$ *and* $0 < \alpha \leqslant 1$. *Suppose there exists* $M > 0$ *such that*

$$⨍_B |u(x) - \bar{u}_B| \mathrm{d}x \leqslant M r^\alpha$$

for all balls $B \subset \Omega$. *Then* $u \in C^{0,\alpha}(\Omega)$ *and*

$$\operatorname{osc}_{B(x,r/2)} u \leqslant C(n, \alpha) M r^\alpha.$$

Here,

$$|B(x,r)| = \frac{\omega_n}{n} r^n,$$

$$\bar{u}_{B(x,r)} = \frac{1}{|B|} \int_B u(y) \mathrm{d}y = ⨍ u(y) \mathrm{d}y,$$

$$\operatorname{osc}_B u = \sup_{x,y \in B} (u(x) - u(y)) = \sup_{x,y \in B} |u(x) - u(y)|.$$

and finally $C^{0,\alpha}$ *is the space of Hölder-continuous functions with exponent* α.

Proof. Let x be a Lebesgue point of u. Suppose $B(x, r/2) \subset B(z, r) \subset \Omega$. Then

$$\begin{aligned} |\bar{u}_{B(x,r/2)} - \bar{u}_{B(z,r)}| &= \left| \frac{1}{|B(x,r/2)|} \int_{B(x,r/2)} u(y) - \bar{u}_{B(z,r)} \mathrm{d}y \right| \\ &\leqslant \frac{1}{|B(x,r/2)|} \int_{B(x,r/2)} |u - \bar{u}_{B(z,r)}| \mathrm{d}y \\ &\leqslant \frac{1}{|B(x,r/2)|} \int_{B(z,r)} |u - \bar{u}_{B(z,r)}| \mathrm{d}y \\ &\leqslant 2^n ⨍_{B(z,r)} |u - \bar{u}_{B(z,r)}| \mathrm{d}y \leqslant 2^n \cdot M r^\alpha. \end{aligned}$$

Choose $z = x$ and iterate this inequality for increasingly smaller balls. This yields

$$\begin{aligned} \left| \bar{u}_{B(x,r/2^k)} - \bar{u}_{B(x,r)} \right| &\leqslant 2^n M \sum_{i=1}^k \left(\frac{r}{2^i} \right)^\alpha \\ &\leqslant C M r^\alpha \end{aligned}$$

independent of k. Since x is a Lebesgue point,

$$\lim_{k \to \infty} \bar{u}_{B(x,r/2^k)} = u(x).$$

Thus

$$|u(x)-\bar{u}_{B(x,r/2)}| \leqslant C(n,\alpha) M r^{\alpha},$$

which also yields

$$\begin{aligned} |u(x)-\bar{u}_{B(z,r)}| &\leqslant |u(x)-\bar{u}_{B(x,r/2)}| + |\bar{u}_{B(x,r/2)}-\bar{u}_{B(z,r)}| \\ &\leqslant C(n,\alpha) M r^{\alpha}. \end{aligned}$$

For any Lebesgue points x, y s.t.

$$B(x,r/2) \subset B(z,r) \quad \text{and} \quad B(y,r/2) \subset B(z,r),$$

this inequality holds:

$$|u(x)-u(y)| \leqslant C(n,\alpha) M r^{\alpha}.$$

This shows $u \in C^{0,\alpha}$. □

3.2 Poincaré's and Morrey's Inequality

To obtain Poincaré's and Morrey's Inequalities, first consider some potential estimates. Consider the Riesz kernels

$$I_{\alpha}(x) = |x|^{\alpha-n}$$

for $0<\alpha<n$ and the Riesz potential

$$(I_{\alpha} * f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \mathrm{d}y.$$

In $\mathbb{R}^n$, $|x|^{\alpha-n} \in L^1_{\text{loc}}$, for $0<\alpha<n$, but not $\alpha=0$.

Lemma 3.14. *Suppose* $0<|\Omega|<\infty$, $0<\alpha<n$. *Then*

$$\int_{\Omega} |x-y|^{\alpha-n} \mathrm{d}y \leqslant C(n,\alpha) |\Omega|^{\alpha/n},$$

where

$$C(n,\alpha) = \omega_n^{1-\alpha/n} \frac{n^{\alpha/n}}{\alpha}.$$

Proof. Let $x \in \Omega$, without loss $x=0$. choose $B(0,r)$ with $r>0$ such that $|B(0,r)| = |\Omega|$

$$\begin{aligned} \int_{\Omega} |y|^{\alpha-n} \mathrm{d}y &= \int_{\Omega\cap B} |y|^{\alpha-n} \mathrm{d}y + \int_{\Omega\setminus B} |y|^{\alpha-n} \mathrm{d}y, \\ \int_{B} |y|^{\alpha-n} \mathrm{d}y &= \int_{\Omega\cap B} |y|^{\alpha-n} \mathrm{d}y + \int_{B\setminus\Omega} |y|^{\alpha-n} \mathrm{d}y. \end{aligned}$$

We know

$$\begin{aligned} \int_{\Omega\setminus B} |y|^{\alpha-n} \mathrm{d}y &\leqslant r^{\alpha-n} \int_{\Omega\setminus B} 1 \mathrm{d}y \\ &= r^{\alpha-n} \int_{B\setminus\Omega} 1 \mathrm{d}y \\ &\leqslant \int_{B\setminus\Omega} |y|^{\alpha-n} \mathrm{d}y \end{aligned}$$

Thus,

$$\int_{\Omega} |y|^{\alpha-n} \mathrm{d}y \leqslant \int_{B} |y|^{\alpha-n} \mathrm{d}y = \omega_n \int_0^r \rho^{\alpha-n} \rho^{n-1} \mathrm{d}\rho = \frac{\omega_n}{\alpha} r^{\alpha}.$$

Then

$$\frac{\omega_n}{\alpha} r^n \Rightarrow r = \left(\frac{n|\Omega|}{\omega_n}\right)^{1/n}.$$

So,

$$\frac{\omega_n}{\alpha} r^{\alpha} = \frac{w^{1-\alpha/n} n^{\alpha/n}}{\alpha} |\Omega|^{\alpha/n}.$$

□

Theorem 3.15. *Let $1 \leqslant p < \infty$. Suppose $|\Omega| < \infty$ and $f \in L^p(\Omega)$. Then,*

$$\|I_1 f\|_{L^p(\Omega)} \leqslant C_1 \|f\|_{L^p(\Omega)},$$

where

$$C_1 = \omega_n^{1-1/n} n^{1/n} |\Omega|^{1/n}.$$

Recall

$$I_1 f(x) = \int_{\Omega} \frac{f(y)}{|x-y|^{n-1}} \mathrm{d}y, \quad x \in \Omega.$$

Proof. By Lemma 3.14,

$$\int_{\Omega} |x-y|^{1-n} \mathrm{d}y \leqslant C_1.$$

Therefore

$$\begin{aligned} |I_1 f(x)| &\leqslant \int_{\Omega} \frac{|f(y)|}{|x-y|^{n-1}} \mathrm{d}y \leqslant \left(\int_{\Omega} \frac{|f(y)|^p}{|x-y|^{n-1}} \mathrm{d}y \right)^{1/p} \left(\int_{\Omega} \frac{1}{} \right)^{1-1/p} \\ &\leqslant C^{1-1/p} \left(\int_{\Omega} \frac{|f(y)|^p}{|x-y|^{n-1}} \right)^{1/p}. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{\Omega} |I_1 f(x)|^p \mathrm{d}x &\leqslant C_1^{p-1} \int_{\Omega} \int_{\Omega} \frac{|f(y)|^p}{|x-y|^{n-1}} \underbrace{\mathrm{d}y \, \mathrm{d}x}_{\text{flip}} \\ &\leqslant C_1^{p-1} \|f\|_{L^p}^p C^1 \\ &= C_1^p \|f\|_{L^p}^p. \end{aligned}$$

□

Theorem 3.16. (Poincaré's Inequality on convex sets) *Suppose Ω convex, $|\Omega| < \infty$. Let $d = \operatorname{diam}(\Omega)$. Suppose $u \in W^{1,p}(\Omega)$, $1 \leqslant p < \infty$. Then*

$$⨍_{\Omega} |u(x) - \bar{u}_{\Omega}|^p \mathrm{d}x \leqslant C(n,p) d^p ⨍_{\Omega} |D u|^p \mathrm{d}x$$

Remark 3.17. Many inequalities relating oscillation to the gradient are called Poincaré Inequalities.

Remark 3.18. This inequality is not scale invariant. It is of the form

$$\left(⨍_{\Omega} |u(x) - \bar{u}_{\Omega}|^p \mathrm{d}x \right)^{1/p} \leqslant C_{\text{universal}} \cdot \underbrace{d}_{\text{length}} \left(⨍_{\Omega} |D u|^p \mathrm{d}x \right)^{1/p}.$$

Corollary 3.19. (Morrey's Inequality) *Let $u \in W^{1,1}(\Omega)$ and $0 < \alpha \leqslant 1$. Suppose there is $M > 0$ s.t.*

$$\int_{B(x,r)} |D u| \mathrm{d}x \leqslant M r^{n-1+\alpha}$$

for all $B(x,r) \subset \Omega$. Then $u \in C^{0,\alpha}(\Omega)$ and

$$\operatorname{osc}_{B(x,r)} u \leqslant C M r^{\alpha}, \quad C = C(n, \alpha).$$

Proof. For any $B(x,r) \subset \Omega$, Poincaré's Inequality gives

$$⨍ |u - \bar{u}_B| \mathrm{d}x \leqslant C r ⨍_B |D u| = \frac{C r}{(\frac{\omega_n}{n}) r^n} \int_B |D u| \leqslant C M r^{\alpha}.$$

Then use Campanato's Inequality. □

Proof. (of Theorem 3.16) *Step 1.* Using pure calculus, derive

$$|u(x)-\bar{u}\,|\leqslant\frac{d^n}{n}\fint_{\Omega}\frac{|Du(y)|}{|x-y|^{n-1}}\mathrm{d}y.$$

Let $|\omega|=1$ and

$$\delta(\omega)=\sup_{t>0}\{x+t\omega\in\Omega\},$$

which can be seen as the distance to the bounary in the direction ω. Let $y=x+t\omega$ and $0\leqslant t\leqslant\delta(\omega)$. Then

$$\begin{aligned}|u(x)-u(y)| &= |u(x)-u(x+t\omega)|\\ &\leqslant \int_0^t|Du(x+s\omega)|\mathrm{d}s\\ &\leqslant \int_0^{\delta(\omega)}|Du(x+s\omega)|\mathrm{d}s.\end{aligned}$$

Since

$$u(x)-\bar{u}=u(x)-\fint_{\Omega}u(y)\mathrm{d}y=\fint_{\Omega}u(x)-u(y)\mathrm{d}y,$$

we have

$$\begin{aligned}|u(x)-\bar{u}\,| &\leqslant \fint_{\Omega}|u(x)-u(y)|\mathrm{d}y\\ &= \frac{1}{|\Omega|}\int_{S^{n-1}}\int_0^{\delta(\omega)}|u(x)-u(x+t\omega)|t^{n-1}\mathrm{d}t\,\mathrm{d}\omega\\ &\leqslant \frac{1}{|\Omega|}\int_{S^{n-1}}\int_0^{\delta(\omega)}\int_0^{\delta(\omega)}|Du(x+s\omega)|\mathrm{d}s\,t^{n-1}\mathrm{d}t\,\mathrm{d}\omega\\ &\leqslant \frac{1}{|\Omega|}\left(\int_{S^{n-1}}\int_0^{\delta(\omega)}\frac{|Du(x+s\omega)}{s^{n-1}}s^{n-1}\mathrm{d}s\,\mathrm{d}\omega\right)\cdot\frac{d^n}{n},\end{aligned}$$

considering

$$\max_{\omega}\int_0^{\delta(\omega)}t^{n-1}\mathrm{d}t=\max_{\omega}\frac{\delta^n(\omega)}{n}=\frac{d^n}{n}.$$

Rewrite the integral using

$$s^{n-1}\mathrm{d}s\,\mathrm{d}\omega=\mathrm{d}y$$

as

$$|u(x)-\bar{u}\,|\leqslant\frac{d^n}{n}\fint\frac{|Du(y)|}{|x-y|^{n-1}}\mathrm{d}y.$$

Recall that

$$I_1f(x)\overset{\mathrm{def}}{=}\int_{\Omega}\frac{f(y)}{|x-y|^{n-1}}\mathrm{d}y.$$

Using Theorem 3.15 on Riesz potentials, we have

$$\begin{aligned}\int_{\Omega}|u(x)-\bar{u}\,|^p\mathrm{d}x &\leqslant \int_{\Omega}\left(\frac{d^n}{n|\Omega|}\right)^p\left(\int_{\Omega}\frac{|Du(y)|}{|x-y|^{n-1}}\mathrm{d}y\right)^p\mathrm{d}x\\ &\leqslant \left(\frac{d^n}{n|\Omega|}\right)^pC_1^p\int_{\Omega}|Du(y)|^p\mathrm{d}y\end{aligned}$$

with $C_1=\omega_n^{1-1/n}n^{1/n}|\Omega|^{1/n}$. Thus

$$\|u-\bar{u}\,\|_{L^p(\Omega)}\leqslant\underbrace{\frac{d^n}{n|\Omega|}\omega_n^{1-1/n}n^{1/n}|\Omega|^{1/n}}_{\frac{d^n\omega^{1-1/n}}{(n|\Omega|)^{1-1/n}}=\left(\frac{\omega_nd^n}{n|\Omega|}\right)^{1/n}}\|Du\|_{L^p(\Omega)}$$

Now, realize that $\frac{\omega_nd^n}{n|\Omega|}$ is just the ratio of volumes of ball of diameter d to volume of $|\Omega|$, which is universally bounded by the isoperimetric inequality. So, the inequality takes the form

$$\|u-\bar{u}\,\|_{L^p(\Omega)}\leqslant\underbrace{C(n)}_{\text{universal}}\cdot\underbrace{d}_{\text{length}}\cdot\|Du\|_{L^p(\Omega)}.$$

□

3.3 The Sobolev Inequality

The desire to make Poincaré's Inequality scale-invariant leads to

Theorem 3.20. (Sobolev Inequality) *Suppose $u \in C_c^1(\mathbb{R}^n)$. Then for $1 \leqslant p < n$, we have*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leqslant C(n,p)\|Du\|_{L^p(\mathbb{R}^n)},$$

where

$$p^* = \frac{np}{n-p}.$$

Remark 3.21. This inequality is *scale-invariant*, and p^* is the only allowable exponent. Suppose we had

$$\left(\int_{\mathbb{R}^n} |u(x)|^q \mathrm{d}x\right)^{1/q} \leqslant C(n,p,q)\left(\int_{\mathbb{R}^n} |Du(x)|^p \mathrm{d}x\right)^{1/p}$$

for every $u \in C_c^1(\mathbb{R}^n)$. Then since $u_\alpha(x) = u(x/\alpha)$ for $\alpha > 0$ is also in $\mathbb{R}^n$, we must also have

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} |u_\alpha(x)|^q \mathrm{d}x\right)^{1/q} &\leqslant C(n,p,q)\left(\int_{\mathbb{R}^n} |Du_\alpha(x)|^p \mathrm{d}x\right)^{1/p} \\
\Leftrightarrow \left(\alpha^n \int_{\mathbb{R}^n} |u_\alpha(x)|^q \mathrm{d}\frac{x}{\alpha}\right)^{1/q} &\leqslant C(n,p,q)\left(\frac{1}{\alpha^p}\int_{\mathbb{R}^n} \left|Du\left(\frac{x}{\alpha}\right)\right|^p \mathrm{d}x\right)^{1/p} \\
\Leftrightarrow \left(\alpha^n \int_{\mathbb{R}^n} |u(x)|^q \mathrm{d}x\right)^{1/q} &\leqslant C(n,p,q)\left(\frac{\alpha^n}{\alpha^p}\int_{\mathbb{R}^n} |Du(x)|^p \mathrm{d}x\right)^{1/p}.
\end{aligned}$$

We then have

$$\alpha^{n/q}\|u\|_{L^q} \leqslant \frac{\alpha^{n/p}}{\alpha} C\|Du\|_{L^p}.$$

Unless

$$\alpha^{n/q} = \alpha^{n/p-1},$$

we have contradiction: simply choose $\alpha \to 0$ or $\alpha \to \infty$. So we must have

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{n} \quad \text{or} \quad q = \frac{np}{n-p} \stackrel{\text{def}}{=} p^*.$$

Remark 3.22. Suppose $p = 1$. Then the Inequality is

$$\|u\|_{L^{1^*}(\mathbb{R}^n)} \leqslant C_n \|Du\|_{L^1(\mathbb{R}^n)}.$$

Consider $1^* = \frac{n}{n-1}$. The best constant is when $u = \mathbf{1}_{B(0,1)}$. Then

$$\text{LHS} = \left(\int_{\mathbb{R}^n} \mathbf{1}_{B(0,1)}^{\frac{n}{n-1}}(x)\mathrm{d}x\right)^{\frac{n}{n-1}} = |B|^{\frac{n-1}{n}} = \left(\frac{\omega_n}{n}\right)^{\frac{n-1}{n}}.$$

And,

$$\text{RHS} = \int_{\mathbb{R}^n} |Du(x)|\mathrm{d}x = (n-1)\text{-dimensional volume} = \omega_n.$$

So, we have

$$\left(\frac{\omega_n}{n}\right)^{\frac{n-1}{n}} \leqslant C \cdot \omega_n.$$

This gives the sharp constant. Thus it turns out that in this case the Sobolev Inequality is nothing but the Isoperimetric Inequality.

Proof.

$$u(x) = \int_{-\infty}^{x} D_k u(\underbrace{x_1, \ldots, x_{k-1}, y_k, x_{k+1}, \ldots, x_n}_{\text{Notation: } \hat{x}_k :=})\mathrm{d}y_k. \tag{3.1}$$

Then

$$|u(x)| \leqslant \int_{\mathbb{R}} |D_k u(\hat{x}_k)\mathrm{d}y_k, \quad k = 1, \ldots, n.$$

First assume $p=1$, $p^*=1^*=n/(n-1)$, $n>1$. Then

$$|u(x)|^{n/(n-1)} \leqslant \prod_{k=1}^{n} \left(\int_{\mathbb{R}} |D_k u(\hat{x}_k)| \mathrm{d}y_k \right)^{1/(n-1)}.$$

We need a *generalized Hölder Inequality:*

$$\int_{\mathbb{R}} f_1 f_2 \cdots f_m \mathrm{d}x \leqslant \|f_1\|_{p_1} \|f_2\|_{p_2} \cdots \|f_m\|_{p_m},$$

provided

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = 1.$$

In particular, we have

$$\int_{\mathbb{R}} f_2^{1/(n-1)} f_3^{1/(n-1)} \cdots f_n^{1/(n-1)} \mathrm{d}x \leqslant \left(\int_{\mathbb{R}} f_2 \right)^{1/(n-1)} \cdots \left(\int_{\mathbb{R}} f_m \right)^{1/(n-1)},$$

choosing $p_2 = p_3 = \cdots p_n = n-1$. Progressively integrate (3.1) on $x_1, \ldots, x_n$ and apply Hölder's Inequality.

Step 1:

$$\begin{aligned}
\int_{\mathbb{R}} |u(x)|^{n/(n-1)} \mathrm{d}x_1 &\leqslant \underbrace{\left(\int_{\mathbb{R}} |D_1 u(\hat{x}_1)| \mathrm{d}y_1 \right)^{1/(n-1)}}_{\text{doesn't depend on } x_1} \cdots \int_{\mathbb{R}} \prod_{k=2}^{n} \Bigg(\underbrace{\int_{\mathbb{R}} D_k u(\hat{x}_k) \mathrm{d}y_k}_{\text{treat as } f_k(x_1)} \Bigg)^{1/(n-1)} \mathrm{d}x_1 \\
&\leqslant \left(\int_{\mathbb{R}} |D_1 u(\hat{x}_1)| \mathrm{d}y_1 \right)^{1/(n-1)} \prod_{k=2}^{n} \left(\int_{\mathbb{R}} \int_{R} |D_k u(\hat{x}_k)| \mathrm{d}y_k \mathrm{d}x_1 \right)^{1/(n-1)}.
\end{aligned}$$

Step 2: Now integrate over x_2:

$$\begin{aligned}
\int_{\mathbb{R}} \int_{\mathbb{R}} |u(x)|^{\frac{n}{n-1}} \mathrm{d}x_1 \mathrm{d}x_2 &\leqslant \underbrace{\left(\int_{\mathbb{R}} \int_{\mathbb{R}} |D_2 u(\hat{x}_2)| \mathrm{d}x_1 \mathrm{d}y_2 \right)^{\frac{1}{n-1}}}_{\text{doesn't see } x_2} \\
&\quad \times \int_{\mathbb{R}} \left(\int_{\mathbb{R}} D_1 u(\hat{x}_1) \mathrm{d}y_1 \right)^{\frac{1}{n-1}} \prod_{k=3}^{n} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |D_k u(\hat{x}_k)| \mathrm{d}y_k \mathrm{d}x_1 \right)^{\frac{1}{n-1}} \mathrm{d}x_2
\end{aligned}$$

Use Hölder's Inequality again. Repeat this process n times to find

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}} \mathrm{d}x \leqslant \prod_{k=1}^{n} \left(\int_{\mathbb{R}^n} |D_k u| \mathrm{d}x \right)^{\frac{1}{n-1}}$$

or

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}} \mathrm{d}x \right)^{\frac{n-1}{n}} &\leqslant \prod_{k=1}^{n} \left(\int_{\mathbb{R}^n} |D_k u| \mathrm{d}x \right)^{\frac{1}{n}} \\
&\leqslant \sum_{k=1}^{n} \frac{1}{n} \int_{\mathbb{R}^n} |D_k u| \mathrm{d}x,
\end{aligned}$$

where we used

$$\sqrt[n]{a_1 \cdots a_n} \leqslant \frac{a_1 + \cdots + a_n}{n}.$$

Since

$$|Du| = \sqrt{|D_1 u|^2 + \cdots + |D_n u|^2},$$

we have by Cauchy-Schwarz

$$\frac{1}{n} \sum_{k=1}^{n} |D_k u| \leqslant \frac{1}{\sqrt{n}} |Du|.$$

Therefore,

$$\boxed{\|u\|_{1^*} \leqslant \frac{1}{\sqrt{n}} \|Du\|_{L^1}.}$$

For $p \neq 1$, we use the fact that

$$Du^\gamma = \gamma u^{\gamma - 1} Du$$

for any γ. Therefore we may apply the Sobolev Inequality with $p = 1$ to find

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} |u|^{\gamma \cdot \frac{n}{n-1}} \mathrm{d}x \right)^{\frac{n-1}{n}} &\leqslant \frac{1}{\sqrt{n}} \int_{\mathbb{R}^n} |Du^\gamma| \mathrm{d}x = \frac{\gamma}{\sqrt{n}} \int_{\mathbb{R}^n} |u|^{\gamma - 1} |Du| \mathrm{d}x \\
&\leqslant \frac{\gamma}{\sqrt{n}} \left(\int_{\mathbb{R}^n} |u|^{(\gamma-1)p'} \mathrm{d}x \right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}^n} |Du|^p \mathrm{d}x \right)^{\frac{1}{p}}.
\end{aligned}$$

Choose γ that

$$\gamma \cdot \frac{n}{n-1} = (\gamma - 1)\, p'.$$

This works for $1 \leqslant p < n$ and yields

$$\|u\|_{L^{p^*}} \leqslant \frac{n-1}{n^{3/2}} p^* \|Du\|_{L^p},$$

where

$$p^* = \frac{n\,p}{n-p} \to \infty$$

as $p \to n$. □

Theorem 3.23. (Morrey's Inequality) *Suppose* $u \in W^{1,p}(\mathbb{R}^n)$, $n < p \leqslant \infty$. *Then* $u \in C^{0,1-n/p}_{\mathrm{loc}}(\mathbb{R}^n)$. *And*

$$\operatorname{osc}_{B(x,r)} u \leqslant C\,r^{1-n/p} \|Du\|_{L^p}.$$

In particular, if $p = \infty$, u *is locally Lipschitz.*

Proof. Poincaré's Inequality in $W^{1,1}_{\mathrm{loc}}$ reads

$$\fint_{B(x,r)} |u - \bar{u}_B| \mathrm{d}x \leqslant C\,r \fint_{B(x,r)} |Du| \mathrm{d}x.$$

Therefore, by Jensen's Inequality

$$\begin{aligned}
\fint_{B(x,r)} |u - \bar{u}_B| \mathrm{d}x &\leqslant C\,r \left(\fint_{B(x,r)} |Du|^p \mathrm{d}x \right)^{1/p} \\
&= C\,r \frac{1}{\left(\frac{\omega_n}{n} r^n \right)^{1/p}} \|Du\|_{L^p(B)} \\
&\leqslant C\,r^{1-n/p} \|Du\|_{L^p}.
\end{aligned}$$

Now apply Campanato's Inequality. □

3.4 Imbeddings

What have we obtained?

$$\begin{tikzcd}
 & L^{p^*} \text{ for } 1 \leqslant p < n \\
W^{1,p}(\mathbb{R}^n) \arrow[ur] \arrow[r] \arrow[dr] & ? \text{ for } p = n \\
 & C^{0,1-\frac{n}{p}}_{\mathrm{loc}}(\mathbb{R}^n) \text{ for } n < p \leqslant \infty.
\end{tikzcd}$$

Figure 3.1.

Typical example where we need $W^{1,n}$: Suppose u is a map $\mathbb{R}^n \to \mathbb{R}^n$. (We are often interested in $\det(Du)$.) Especially care about

$$\int_{\Omega} \det(Du) \mathrm{d}x$$

for $\Omega \subset\subset \mathbb{R}^n$. Then

$$\det(Du) = \sum_{\sigma} (-1)^{\sigma} u_{1,\sigma_1} \cdots u_{n,\sigma_n}$$

So, we need $u_{i,j} \in L^n(\Omega)$ or $u \in W^{1,n}$.

Theorem 3.24. (John-Nirenberg) *If $u \in W^{1,n}(\mathbb{R}^n)$, then $u \in \mathrm{BMO}(\mathbb{R}^n)$, where*

$$[u]_{\mathrm{BMO}} = \sup_B \fint_B |u - \bar{u}_B| \mathrm{d}x$$

and $\mathrm{BMO}(\mathbb{R}^n) := \{[u]_{\mathrm{BMO}} < \infty\}$.

For a compact domain,

$$L^1 \to \mathcal{H}^1 \quad L^p \subset \cdots \subset L^{\infty} \subset \mathrm{BMO},$$

where $\mathcal{H}^1$ is contained in the dual of BMO.

Definition 3.25. *A Banach space B_1 is* imbedded *into a Banach space B_2 (written $B_1 \to B_2$) if there is a continuous, linear one-to-one mapping $T: B_1 \to B_2$.*

Example 3.26. $W^{1,p}(\mathbb{R}^n) \to L^{p^*}(\mathbb{R}^n)$ for $1 \leqslant p < n$.

Let Ω be bounded.

Example 3.27. $W_0^{1,p}(\Omega) \to C^{0,1-n/p}(\bar{\Omega})$ for $n < p \leqslant \infty$.

Example 3.28. $W_0^{1,p}(\Omega) \to L^q(\Omega)$ for $1 < p < n$ and $1 \leqslant q < p^*$, where we used

$$\|u\|_{L^q(\Omega)} \leqslant \|u\|_{L^{p^*}} |\Omega|^{1-q/p^*},$$

which is derived from Hölder's Inequality.

Definition 3.29. *The imbedding is* compact *(written $B_1 \hookrightarrow B_2$) if the image of every bounded set in B_1 is precompact in B_2.*

Recall that in a complete metric: precompact $\Leftrightarrow$ totally bounded.

Theorem 3.30. (Rellich-Kondrachev) *Assume Ω is bounded. Then*

1. *$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leqslant p < n$ and $1 \leqslant q < p^*$.*
2. *$W_0^{1,p}(\Omega) \hookrightarrow C^0(\bar{\Omega})$ for $n < p \leqslant \infty$.*

Remark 3.31. We only have strict inequality in part 1. (That is, $q = p^*$ does not work.)

Proof. *Of part 2:* By Morrey's Inequality, $W^{1,p}(\Omega) \to C^{0,1-n/p}(\bar{\Omega})$. Now apply the Arzelà-Ascoli theorem.

Of part 1: We have to reduce to Arzelà-Ascoli. Let A be a bounded set in $W_0^{1,p}(\Omega)$. We may as well assume that $A \subset C_c^1(\Omega)$. Let $\psi \geqslant 0$ be a standard mollifier. Consider the family

$$A_{\varepsilon} = \{u * \psi_{\varepsilon} | u \in A\}, \quad \psi_{\varepsilon}(y) = \frac{1}{\varepsilon^n} \psi\left(\frac{y}{\varepsilon}\right).$$

Claim: A_{ε} is precompact in $C^0(\bar{\Omega})$.

Proof: We must show A_{ε} is uniformly bounded, equicontinuous.

$$u_{\varepsilon}(x) = \frac{1}{\varepsilon^n} \int_{\Omega} \psi\left(\frac{x-y}{\varepsilon}\right) u(y) \mathrm{d}y = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} \psi\left(\frac{x-y}{\varepsilon}\right) u(y) \mathrm{d}y.$$

Therefore,

$$\begin{aligned}|u_\varepsilon(x)| &\leqslant \frac{\|\psi\|_\infty}{\varepsilon^n}\|u\|_{L^1(\Omega)}.\\ &\leqslant \frac{\|\psi\|_\infty}{\varepsilon^n}|\Omega|^{1-1/p}\|u\|_{L^p(\Omega)}\\ &\leqslant \frac{M\|\psi\|_\infty}{\varepsilon^n}|\Omega|^{1-1/p}.\end{aligned}$$

Similarly,

$$Du_\varepsilon(x)=\frac{1}{\varepsilon^{n+1}}\int_{\mathbb{R}^n} D\psi\left(\frac{x-y}{\varepsilon}\right)u(y)\mathrm{d}y.$$

Thus

$$|Du_\varepsilon(x)|\leqslant\frac{M}{\varepsilon^{n+1}}\|D\psi\|_\infty|\Omega|^{1-1/p}.$$

The claim is thereby established.

In particular, the claim implies A_ε is precompact in $L^1(\Omega)$. (Indeed, if u_ε^k is convergent in $C^0(\bar{\Omega})$, then by DCT, u_ε^k is convergent in $L^1(\Omega)$.

We also have the estimate

$$\begin{aligned}|u(x)-u_\varepsilon(x)| &= \left|\frac{1}{\varepsilon^n}\int_{\mathbb{R}^n}\psi\left(\frac{y}{\varepsilon}\right)(u(x)-u(x-y))\mathrm{d}y\right|\\ &\overset{z=y/\varepsilon,\operatorname{supp}(\psi)\subset B(0,1)}{=} \left|\int_{B(0,1)}\psi(z)(u(x)-u(x-\varepsilon z))\mathrm{d}z\right|\end{aligned}$$

By the fundamental theorem of calculus, the subterm

$$u(x)-u(x-\varepsilon z)=\int_0^1\frac{\mathrm{d}}{\mathrm{d}t}u(x-\varepsilon\, t\, z)\mathrm{d}t\leqslant\int_0^1 Du(x-\varepsilon t\, z)\cdot z\,\mathrm{d}t.$$

Then

$$|u(x)-u_\varepsilon(x)| \leqslant \int_{B(0,1)}\psi(z)\int_0^{\varepsilon|z|}|Du(x-t\omega)|\mathrm{d}t\,\mathrm{d}z,\quad \omega=\frac{z}{|z|}.$$

(We use $\psi\geqslant 0$ and differentiability on a line.) Therefore,

$$\begin{aligned}\int_\Omega|u(x)-u_\varepsilon(x)|\mathrm{d}x &\leqslant \int_{B(0,1)}\psi(z)\int_0^{\varepsilon|z|}\underbrace{\int_\Omega|Du(x-t\omega)|\mathrm{d}x}_{(*)}\mathrm{d}t\,\mathrm{d}z\\ &\leqslant \|Du\|_{L^1(\Omega)}\int_{B(0,1)}\psi(z)\int_0^{\varepsilon|z|}\mathrm{d}t\,\mathrm{d}z\\ &\leqslant \varepsilon\|Du\|_{L^1(\Omega)}\leqslant\varepsilon M|\Omega|^{1-1/p},\end{aligned}$$

where

$$(*)=\int_\Omega|Du(x-t\omega)|\mathrm{d}x\leqslant\int_\Omega|Du(x)|\mathrm{d}x.$$

using $u\in C_c^1+$zero extension. Summary:

- A_ε precompact in $L^1(\Omega)\Leftrightarrow$totally bounded,
- Every $u\in A$ is ε-close to $u_\varepsilon\in A_\varepsilon$.

Therefore A is totally bounded in L^1.

This shows that A is precompact in $L^1(\Omega)$. If $1\leqslant q<p^*$, we have

$$\|u-u_\varepsilon\|_{L^q}\leqslant\|u-u^\varepsilon\|_{L^1(\Omega)}\|u-u^\varepsilon\|_{L^{p^*}(\Omega)}\leqslant\underbrace{\varepsilon^\theta}_{\text{just proved}}\cdot\underbrace{(2M)^{1-\theta}}_{\text{Sobolev's}},$$

where

$$\frac{1}{q}=\frac{\theta}{1}+\frac{1-\theta}{p^*}.$$

Therefore A is totally bounded in $L^q(\Omega)$. ☐

$$
\begin{tikzcd}
\mathrm{BV} \subset W^{1,1} & W^{1,p} \arrow[d] & W^{1,n} \arrow[d] & W^{1,p} \arrow[d] & W^{1,\infty} \arrow[d] \\
& L^{p^*} & \mathrm{BMO} \arrow[d, hook] & C^{0,1-n/p} & \mathrm{Lip}(\Omega) \\
& \text{Also } \hookrightarrow L^q & C^0(\bar{\Omega}) & &
\end{tikzcd}
$$

Figure 3.2.

4 Scalar Elliptic Equations

Reference: Gilbarg/Trudinger, Chapter 3 and 8

The basic setup in divergence form:

$$
\begin{aligned}
Lu &= \operatorname{div}(A\,Du + bu) + c \cdot Du + d\,u \\
&= D_i(a_{i,j} D_j u + b_i u) + c_i D_i u + d\,u,
\end{aligned}
$$

where $A: \Omega \to \mathbb{M}^{n \times n}$, $b, c: \Omega \to \mathbb{R}^n$, $d: \Omega \to \mathbb{R}$. Main assumptions:

1. Strict ellipticity: There exists $\lambda > 0$ such that
$$\xi^T A(x) \xi \geqslant \lambda |\xi|^2$$
for every $x \in \Omega$, $\xi \in \mathbb{R}^n$.
2. $A, b, c, d \in L^\infty(\Omega)$.

There exists $\Lambda > 0$, $\nu > 0$ such that

$$\|A\|_{L^\infty(\Omega)} \stackrel{\text{def}}{=} \left\| \sqrt{\operatorname{Tr}(A^T A)} \right\|_{L^\infty(\Omega)} \leqslant \Lambda$$

and

$$\frac{1}{\lambda} \left(\|b\|_\infty + \|c\|_\infty + \|d\|_\infty \right) \leqslant \nu.$$

Motivation: Typical problem is to minimize

$$I[u] = \int_\Omega E(Du) \mathrm{d}x,$$

where E is "energy". If u is a minimizer, we obtain the Euler-Lagrange equations as follows:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} I[u + t\,v]|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega E(D(u + t\,v)) \mathrm{d}x|_{t=0} = \int_\Omega DE(D(u + t\,v)) \cdot Dv\, \mathrm{d}x \\
&= \int_\Omega DE(D\,u) \cdot Dv\, \mathrm{d}x.
\end{aligned}
$$

Necessary condition for minimum:

$$\int_\Omega DE(Du) \cdot Dv\, \mathrm{d}x = 0$$

for all test functions v. This "means" that

$$\int_\Omega D(DE(Du)) \cdot v\, \mathrm{d}x,$$

which is the term that we had in the first place–namely the *Euler-Lagrange equations*:

$$\operatorname{div}(DE(Du)) = 0$$

with $u\colon\Omega\to\mathbb{R}^n$ and $E\colon\mathbb{R}^n\to\mathbb{R}$ is a given smooth function, for example $E(u)=|Du|^p$ for $p>1$. In coordinates,

$$D_i[D_{p_i}E(D_j u)]=0\Rightarrow D_{p_i,p_j}E(D_j u)\cdot D_{i,j}u=0 \quad \text{or} \quad \operatorname{tr}(A\,D^2u)=0,$$

where $A(x)=D^2E(Du(x))$, which is the unknown as yet.

Regularity problem: Assuming u solves the above problem. Show that u is regular. A priori, we only know that $A\in L^\infty\to$ DeGiorgi and Nash $\Rightarrow$ classical regularity.

4.1 Weak Formulation

Formally multiply $Lu=0$ by $v\in C_c^1(\Omega)$ and integrate by parts:

$$\begin{aligned} & \int_\Omega (\operatorname{div}(A\,Du+b\,u)+(c\cdot Du+d\,u))\cdot v\,\mathrm{d}x \\ = & \int_\Omega (Dv^TA\,Du+b\cdot Dv\,u)+(c\cdot Du+d\,u)v\,\mathrm{d}x \\ =: & \; B[u,v]. \end{aligned}$$

Basic assumption: $u\in W^{1,2}(\Omega)$. Then $B[u,v]$ is well-defined for all $v\in C_c^1(\Omega)$ and by Cauchy-Schwarz for all $v\in W_0^{1,2}(\Omega)$.

Now consider the classical Dirichlet problem:

$$\begin{aligned} Lu &= f \quad \text{on}\,\Omega, \\ u &= g \quad \text{on}\,\partial\Omega. \end{aligned}$$

Definition 4.1. (Generalized Dirichlet Problem) *Given $g\in L^2(\Omega)$, $f\in L^2(\Omega)$, $\varphi\in W^{1,2}(\Omega)$. $u\in W^{1,2}(\Omega)$ is a solution to*

$$\begin{aligned} Lu &= g+\operatorname{div} f \quad in\,\Omega, \\ u &= \varphi \quad on\,\partial\Omega \end{aligned}$$

if

1. $B[u,v]=F[v]:=\int_\Omega [g\,v-f\cdot Dv]\mathrm{d}x$ *for* $v\in C_0^1(\Omega)$
2. $u-\varphi\in W_0^{1,2}(\Omega)$.

4.2 The Weak Maximum Principle

We want $Lu\geqslant 0\Rightarrow \sup_\Omega u\leqslant \sup_{\partial\Omega}u$. *Catch:* How do we define $\sup_{\partial\Omega}u$?

Definition 4.2. *Suppose $u\in W^{1,2}(\Omega)$. We say $u\leqslant 0$ on $\partial\Omega$ if*

$$u^+=\max(u,0)\in W_0^{1,2}(\Omega).$$

Similarly, $u\leqslant v$ on $\partial\Omega$ if

$$(u-v)^+\in W_0^{1,2}(\Omega).$$

Definition 4.3.

$$\sup_{\partial\Omega}u=\inf\{k\in\mathbb{R}\colon u\leqslant k\}=\inf\left\{k\in\mathbb{R}\colon (u-k)^+\in W_0^{1,2}(\Omega)\right\}.$$

Basic assumptions:

(E_1). There is a $\lambda>0$ such that $\xi^TA(x)\xi\geqslant\lambda|\xi|^2$ for all $x\in\Omega$, $\xi\in\mathbb{R}^n$.

(E_2). There is $\Lambda>0$, $\nu>0$ such that

$$\frac{1}{\lambda^2}(\|b\|_\infty+\|c\|_\infty)^2+\frac{1}{\lambda}\|d\|_\infty\leqslant\nu^2, \quad \left\|\operatorname{tr}(A^TA)\right\|_\infty\leqslant\Lambda^2.$$

Definition 4.4. (The Generalized Dirichlet Problem) *Given f, g, φ, find $u \in W^{1,2}(\Omega)$ such that*

$$\begin{array}{rrcll} (*) & Lu & = & g + \operatorname{div} f & \text{in } \Omega, \\ (\#) & u & = & \varphi & \text{on } \partial\Omega, \end{array}$$

where $(*)$ means $B[u,v] = F[v]$ and $(\#)$ means $u - \varphi \in W_0^{1,2}(\Omega)$ with

$$\begin{aligned} B[u,v] &= \int_\Omega Dv^T(A\,Du - b\,u) - (c \cdot Du + b)v\,\mathrm{d}x, \\ F(v) &= \int_\Omega Dv \cdot f - g\,v\,\mathrm{d}x. \end{aligned}$$

Classical Maximum Principle: If L is *not* in divergence form, say

$$0 = AD^2u + b \cdot Du + d\,u,$$

where we need $d \leqslant 0$ to obtain a maximum principle (see Evans or Gilbarg&Trudinger, Chapter 3).

Additional Assumption for Maximum Principle:

(*E*$_3$). $\operatorname{div} b + d \leqslant 0$ in the weak sense, that is

$$\int_\Omega (\operatorname{div} b + d)v\,\mathrm{d}x \leqslant 0 \quad \forall v \in C_c^1(\Omega), v \geqslant 0.$$

Precisely,

$$\int_\Omega d\,v - b \cdot Dv\,\mathrm{d}x \leqslant 0 \quad \forall v \in C_c^1(\Omega), v \geqslant 0.$$

Definition 4.5. *$u \in W^{1,2}(\Omega)$ is a* subsolution *to the Generalized Dirichlet Problem if $B[u,v] \leqslant F(v)$ for all $v \in C_c^1(\Omega)$ with $v \geqslant 0$, which is*

$$Lu \geqslant g + \operatorname{div} f$$

read in a weak sense.

Theorem 4.6. (Weak Maximum Principle) *Suppose $Lu \geqslant 0$ and (E_1), (E_2), (E_3) hold. Then*

$$\sup_\Omega u \leqslant \sup_{\partial\Omega} u^+.$$

Remark 4.7. Recall

$$\begin{aligned} \sup_{\partial\Omega} u^+ &= \inf\{k \in \mathbb{R}: (u^+ - k)^+ \in W_0^{1,2}(\Omega)\} \\ &= \inf\{k \geqslant 0: (u - k)^+ \in W_0^{1,2}(\Omega)\}. \end{aligned}$$

Remark 4.8. There are no assumptions of boundedness or connectedness or smoothness on Ω.

Compare the above theorem with the classical maximum principle for $\Delta u \geqslant 0$.

Corollary 4.9. *$W^{1,2}(\Omega)$ solutions to the Generalized Dirichlet Problem are unique if they exist.*

Remark 4.10. Nonuniqueness of the extension problem. Consider the ball $B(0,1)$ and

$$u(x) = a + (1-a)|x|^{2-n}$$

for $a \in \mathbb{R}^n$.

$$\int |Du(x)|^2 < \infty \quad \Leftrightarrow \quad a = 0, n \geqslant 3.$$

Proof. (of weak maximum principle) Step 1) The inequality (E_3)

$$\int_\Omega (d\,v - Dv \cdot b)\mathrm{d}x \leqslant 0$$

for $v \geqslant 0$, $v \in C_c^1(\Omega)$ holds for all $v \in W_0^{1,1}(\Omega)$ (since by (E_2), $d, b \in L^\infty$).

Step 2) Basic inequality:

$$B[u, v] \leqslant 0$$

for $v \in C_c^1(\Omega)$ and $v \geqslant 0$.

$$\begin{aligned} \int_\Omega Dv^T(A\,Du + b u) - (c \cdot Du + d\,u)v\,\mathrm{d}x &\leqslant 0 \\ \Rightarrow \int_\Omega Dv^T A \cdot Du - (b + c)Du \cdot v &\leqslant \int_\Omega d(u\,v) - b \cdot D(u\,v)\mathrm{d}x \leqslant 0. \end{aligned}$$

Now choose test functions cleverly such that $u\,v \geqslant 0$ and $u\,v \in W_0^{1,1}(\Omega)$.

(applying step 1) But $D(u\,v) = u\,Dv + v\,Du$ holds for $u\,v \in W_0^{1,1}(\Omega)$ and $u\,v \in W_0^{1,1}(\Omega)$ holds for $u \in W^{1,2}(\Omega)$ and $v \in C_c^1(\Omega)$, which is OK. (See the chain rule for $W^{1,p}$ in Evans.)

$$\int_\Omega Dv^T A\,Du\,\mathrm{d}x \leqslant \int (b + c)Du \cdot v\,\mathrm{d}x,$$

provided $u\,v \geqslant 0$, $v \geqslant 0$, $u\,v \in W_0^{1,1}(\Omega)$.

Step 3) Let $l := \sup_{\partial\Omega} u$. Suppose $\sup_\Omega u > l$ (else there is nothing to prove). Choose $l \leqslant k < \sup_\Omega u$ and $v = (u - k)^+$. We know that $v \in W_0^{1,2}(\Omega)$ by the definition of l.

$$l = \sup_{\partial\Omega} u^+ = \inf\{k \geqslant 0 \colon (u - k)^+ \in W_0^{1,2}(\Omega)\}.$$

Assume $l \leqslant k < \sup_\Omega u =: m$, $v := (u - k)^+$. Then

$$Dv = \begin{cases} Du & u > k, \\ 0 & u \leqslant k. \end{cases}$$

And if $\Gamma = \{Dv \neq 0\}$, we have

$$\lambda \int_\Omega |Dv|^2 \mathrm{d}x \overset{\text{strict ellip.}}{\leqslant} \int_\Omega Dv^T A\,Dv\mathrm{d}x \overset{(E_2)+\text{above}}{\leqslant} 2\nu\lambda \int_\Gamma v|Dv(x)|\mathrm{d}x.$$

$$\int_\Omega |Dv|^2 \leqslant 2\nu\left(\int_\Gamma |v|^2 \mathrm{d}x\right)^{1/2}\left(\int_\Omega |Dv|^2 \mathrm{d}x\right)^{1/2}.$$

Thus we obtain

$$\|Dv\|_{L^2(\Omega)} \leqslant 2\nu\|v\|_{L^2(\Omega)}.$$

By Sobolev's Inequality,

$$\|v\|_{L^{2^*}(\Omega)} \leqslant C_n\|Dv\|_{L^2(\Omega)} \leqslant C_n 2\nu\|v\|_{L^2(\Gamma)} \leqslant C_n 2\nu|\Gamma|^{1/n}\|v\|_{L^{2^*}(\Omega)}.$$

Thus

$$|\Gamma| \geqslant \frac{1}{C_n 2\nu} > 0, \tag{4.1}$$

independent of k. Letting $k \to m$, we obtain that $m < \infty$ (else $u \notin W^{1,2}(\Omega)$. Choosing $k = m$, obtain $Dv = 0$ a.e. contradicting (4.1). □

4.3 Existence Theory

Definition 4.11. *A continuous operator $T \colon B_1 \to B_2$, where B_1 and B_2 are Banach spaces, is called* compact *if $T(A)$ is precompact in B_2 for every bounded set $A \subset B_1$.*

Theorem 4.12. (Fredholm Alternative) *Assume $T \colon B \to B$ is linear, continuous and* compact. *Then either*

1. *$(I - T)\,x = 0$ has a solution $x \neq 0$*
   or

2. $(I-T)^{-1}$ *exists and is a bounded linear operator from* $B \to B$.

Read this as "Uniqueness and Compactness $\Rightarrow$ Existence"

Theorem 4.13. (Lax-Milgram) *Let* $B: \mathcal{H} \times \mathcal{H} \to \mathbb{F}$ *be bilinear form on a Hilbert space such that*

1. $|B[u,v]| \leqslant K\|u\|\|v\|$ *for some* $K>0$,
2. $B[u,u] \geqslant k\|u\|^2$ *for some* $k>0$.

Then for every $F \in \mathcal{H}^*$ *there exists a* $g \in \mathcal{H}$ *such that* $B[u,g] = F(u)$ *for every* $u \in \mathcal{H}$.

Assumption 2 above is called *coercivity*.

Proof. 1) Riesz representation theorem. For any $v \in \mathcal{H}$ the map $u \mapsto B[u,\, v]$ defines a bounded linear functional on $\mathcal{H}$. By the Riesz Representation Theorem, there is $Tv \in \mathcal{H}^*$ such that

$$B[u,v] = Tv(u)$$

for every $u \in \mathcal{H}$. Thus we obtain a linear map $\mathcal{H} \to \mathcal{H}^*$, $v \mapsto Tv$.

2) $|Tv(u)| = |B[u,v]| \leqslant K\|u\|\|v\|$, so $\|T\| \leqslant K$. Moreover,

$$k\|v\|^2 \leqslant B[v,v] = Tv(v) \leqslant \|Tv\|\|v\|.$$

Thus

$$0 < k \leqslant \frac{\|Tv\|}{\|v\|} \leqslant K.$$

Claim: T is one-to-one. $Tv = 0 \Rightarrow k\|v\| \leqslant \|Tv\| = 0 \Rightarrow \|v\| = 0$.

Claim: T is onto. If not, there exists $z \neq 0$ such that $T(\mathcal{H}) \perp z$. Now use that $T(\mathcal{H})$ is closed. Choose $v = z$. Then

$$0 = (z, Tz) = Tz(z) \geqslant k\|z\|^2$$

□

Theorem 4.14. *Let* Ω *be bounded, assume* E_1, E_2, E_3. *Then the Generalized Dirichlet Problem has a solution for every* $f, g \in L^2(\Omega)$ *and* $\varphi \in W^{1,2}(\Omega)$.

Then the Generalized Dirichlet Problem can be stated as finding a $u \in W_0^{1,2}(\Omega)$ such that

$$B[u,v] = F(v) \quad \text{for every } v \in W_0^{1,2}(\Omega).$$

using

$$F(v) \;=\; \int_\Omega (f \cdot Dv - g v)\mathrm{d}x.$$

Proof. (Step 1) Reduce to the case $\varphi = 0$. Consider $\tilde{u} = u - \varphi$.

(Step 2)

Lemma 4.15. (Coercivity) *Assume* (E_1), (E_2) *hold. Then*

$$B[u,u] \geqslant \frac{\lambda}{2}\int_\Omega |Du|^2 - \lambda\nu^2 \int_\Omega |u|^2 \mathrm{d}x.$$

Proof.

$$\begin{aligned}
B[u,u] &= \int_\Omega \underbrace{Du^t[A \cdot Du}_{(1)} + \underbrace{bu] - [c \cdot Du}_{(2)} + \underbrace{d\,u]}_{(3)} u \mathrm{d}x. \\
(1) &= \int_\Omega Du^t A\, Du \mathrm{d}x \underset{(E_1)}{\geqslant} \lambda \int_\Omega |Du|^2 \mathrm{d}x. \\
(2) &\leqslant (\|b\|_\infty + \|c\|_\infty) \int_\Omega |u|\,|Du| \mathrm{d}x \leqslant \frac{\lambda}{2}\|Du\|_{L^2(\Omega)}^2 + \frac{1}{2\lambda}(\|b\|_\infty + \|c\|_\infty)^2 \|u\|_{L^2(\Omega)}^2
\end{aligned}$$

using the elementary inequality

$$2a\,b \leqslant \lambda a^2 + \frac{b^2}{\lambda}$$

for $\lambda > 0$. By assumption (E_2),

$$\frac{\|b\|_\infty^2 + \|c\|_\infty^2}{2\lambda} + \frac{\|d\|_\infty}{2} \leqslant \lambda\nu^2.$$

Now combine these estimates. □

Notation: $\mathcal{H} := W_0^{1,2}(\Omega)$, a Hilbert space. $\mathcal{H}^* = $ dual of $\mathcal{H}$.

Aside: Isn't $\mathcal{H}^* = \mathcal{H}$ by reflexivity of Hilbert spaces? No, only $\mathcal{H} \hat{=} \mathcal{H}^*$. In $\mathbb{R}^n$, we denote

$$H^s(\mathbb{R}^n) := \left\{ u \in \mathcal{S}' \colon \int (1 + |k^2|)^{s/2} |\hat{u}(\xi)|^2 \mathrm{d}\xi < \infty \right\}.$$

This works for every $s \in \mathbb{R}$. If $s = 1$, we have

$$\int_{\mathbb{R}^n} (1 + |k^2|)^{1/2} |\hat{u}(\xi)|^2 \mathrm{d}\xi = C_n \int_{\mathbb{R}^n} (|u|^2 + |Du|^2)\mathrm{d}x = C_n \|u\|_{W^{1,2}(\Omega)}^2.$$

By Parseval's Equation

$$\int_{\mathbb{R}^n} u(x) v^*(x)\mathrm{d}x = C_n \int_{\mathbb{R}^n} \hat{u}(k)\hat{v}^*(k)\mathrm{d}k.$$

If $u \in H^s$, $v \in H^{-s}$, then RHS is

$$(u, v)_{L^2} = \int_{\mathbb{R}^n} (1 + |k|^2)^{s/2} \hat{u}(k) (1 + |k|^2)^{-s/2} \hat{v}^*(k)\mathrm{d}k \leqslant \|u\|_{H^s} \|v\|_{H^{-s}}$$

by Cauchy-Schwarz. (cf. a 1-page paper by Meyer-Serrin??, PNAS, 1960s, the title is $H = W$.) *End aside.*

Every $u \in \mathcal{H}$ also defines an element of $\mathcal{H}^*$ as follows: Define

$$I(u)(v) = \int_\Omega u(x) v(x)\mathrm{d}x \quad \text{for every } v \in H.$$

Recall that the first step in the proof of our Theorem is to reduce to $\varphi = 0$ by setting $\tilde{u} = u - \varphi$ if $\varphi \neq 0$.

Lemma 4.16. (Compactness) *$\mathcal{I}\colon \mathcal{H} \to \mathcal{H}^*$ is compact.*

Proof. $I = I_1 I_2$, where $I_2\colon \mathcal{H} \to L^2$ is compact by Rellich and $I_1\colon L^2 \to \mathcal{H}^*$ is continuous. □

We are trying to solve

$$L u = \underbrace{g + \operatorname{div} f}_{\in \mathcal{H}^*} \tag{4.2}$$

Indeed, given g, f, we have defined

$$F(v) = \int_\Omega (Dv \cdot f - g v)\mathrm{d}x.$$

We treat (4.2) as an equation in $\mathcal{H}^*$. Define

$$L_\sigma = L - \sigma I$$

for $\sigma \in \mathbb{R}$ and the associated bilinear form

$$B_\sigma[u, v] = B[u, v] + \sigma \int_\Omega u(x) v(x)\mathrm{d}x.$$

Thus,

$$\begin{array}{rcl} B_\sigma[u, u] & = & B[u, u] + \sigma \int_\Omega u(x) v(x)\mathrm{d}x \\ & \overset{\text{Lemma 4.15}}{\geqslant} & \frac{\lambda}{2} \int_\Omega |Du|^2 \mathrm{d}x - \lambda\nu^2 \int_\Omega |u|^2 \mathrm{d}x + \sigma \int_\Omega |u|^2 \mathrm{d}x \\ & \geqslant & \frac{\lambda}{2} \left[\int_\Omega |Du|^2 \mathrm{d}x + \int_\Omega |u|^2 \mathrm{d}x \right] = \lambda \|u\|_{\mathcal{H}}^2. \\ \sigma & \geqslant & \lambda\nu^2 + \lambda/2. \end{array}$$

So B_σ is coercive $\Rightarrow$ Lax-Milgram: $L_\sigma^{-1}\colon \mathcal{H}^* \to \mathcal{H}$ is bounded.

$$\begin{aligned}
& Lu = g + \operatorname{div} f \quad \text{in } \mathcal{H}^* \\
\Leftrightarrow\ & L_\sigma u + \sigma I(u) = g + \operatorname{div} f \quad \text{in } \mathcal{H}^*. \\
\Leftrightarrow\ & u + \sigma \underbrace{\underbrace{L_\sigma^{-1}}_{\text{continuous}} \ \underbrace{I(u)}_{\text{compact}}}_{\text{compact}} = L_\sigma^{-1}(g + \operatorname{div} f) \quad \text{in } \mathcal{H}.
\end{aligned}$$

Weak maximum principle $\Rightarrow$ if $g = 0$, $f = 0$, then $u = 0$. By the Fredholm alternative, using $T = L_\sigma^{-1} I \Rightarrow \exists! u$ for every $g + \operatorname{div} f$. $\square$

Remark 4.17. L_σ^{-1} is the abstract Green's function.

4.4 Elliptic Regularity

- Bootstrap arguments: Finite differences and Sobolev spaces
- Weak Harnack Inequalities: Measurable $\to$ Hölder continuous (deGiorgi, Nash, Moser)

4.4.1 Finite Differences and Sobolev Spaces

Let

$$\Delta_i^h u = \frac{u(x + h\, e_i) - u(x)}{h},$$

where e_i is the ith coordinate vector w.r.t. the standard basis of $\mathbb{R}^n$. $\Delta^h u$ is well-defined on $\Omega' \subset\subset \Omega$ provided $h < \operatorname{dist}(\Omega', \partial\Omega)$.

Theorem 4.18. *$\Omega' \subset\subset \Omega$, $h < \operatorname{dist}(\Omega', \partial\Omega)$,*

a) Let $1 \leqslant p \leqslant \infty$ and $u \in W^{1,p}(\Omega)$. Then $\Delta^h u \in L^p(\Omega')$ and

$$\left\| \Delta^h u \right\|_{L^p(\Omega')} \leqslant \| D u \|_{L^p(\Omega)}.$$

b) Let $1 < p \leqslant \infty$. Suppose $u \in L^p(\Omega)$ and

$$\left\| \Delta^h u \right\|_{L^p(\Omega')} \leqslant M,$$

for all $h < \operatorname{dist}(\Omega', \partial\Omega) \Rightarrow u \in W^{1,p}(\Omega')$ and $\| D u \|_{L^p(\Omega')} \leqslant M$.

Ell. regularity started over.

Goal: Existence of weak solutions + smoothness of A, b, c, d, f, g

- $\Rightarrow$ Regularity of weak solutions
- $\Rightarrow$ Uniqueness of classical solutions+Existence.

Basic assumptions: E_1, E_2, E_3 as before, $Lu = g + \operatorname{div} f$ (assume $f = 0$).

Theorem 4.19. *Assume $Lu = g$, E_1 E_2, E_3. Moreover, assume A, b Lipschitz functions. Then for any $\Omega' \subset\subset \Omega$ we have*

$$\| u \|_{W^{2,2}(\Omega')} \leqslant C\Big(\| u \|_{W^{1,2}(\Omega)} + \| g \|_{L^2(\Omega)} \Big),$$

where $C = C(n, \lambda, d', K)$, where $K = \max\left(\operatorname{Lip}(A), \operatorname{Lip}(b), \| c \|_\infty, \| d \|_\infty\right)$ and $d' = \operatorname{dist}(\Omega', \partial\Omega)$. In particular, $Lu = g$ a.e. in Ω.

Proof. Uses finite differences Δ_k^h for $0 < |h| < d'$. It suffices to show $\left\| \Delta_k^h D_i u \right\|_{L^2(\Omega')}$ uniformly bounded for $0 < |h| < d'/2$.

Definition of weak solutions is: for every $v \in C_c^1(\Omega)$

$$\int_\Omega \left[D v^T (A\, D u + b\, u) - (c \cdot D u + d\, u) v \right] \mathrm{d}x = \int_\Omega g\, v\, \mathrm{d}x.$$

Rewrite as

$$\int_\Omega Dv^T(A\,Du)\mathrm{d}x = \int_\Omega \tilde{g}\,v\,\mathrm{d}x \tag{4.3}$$

for all $v\in C_c^1(\Omega)$, where

$$\tilde{g} = g + (c+b)\cdot Du + d\,u.$$

By (E_2) we know that $\tilde{g}_2\in L^2(\Omega)$. Now think about "discrete integration by parts":

$$\int_\Omega (\Delta_k^h v) f(x)\mathrm{d}x \;=\; -\int_\Omega v(x)\Delta_k^{-h} f(x)\mathrm{d}x$$

for every $f\in L^2(\Omega)$. We may replace $v\in C_c^1(\Omega)$ by $\Delta_k^h v\in C_c^1(\Omega)$ in (4.3), provided $0<h<d'/2$. Then we have

$$\int_\Omega Dv^T\underbrace{\Delta_k^h(A\cdot Du)}_{(*)}\mathrm{d}x = -\int_\Omega (D\Delta_k^{-h}v)^T A\,Du\mathrm{d}x \overset{(*)}{=} -\int_\Omega \tilde{g}\,\Delta_k^{-h} v\,\mathrm{d}x. \tag{4.4}$$

In coordinates, $(*)$ is

$$\begin{aligned}\Delta_k^h(a_{i,j}(x)D_j u(x)) &= \frac{a_{i,j}(x+h\,e_k)D_j u(x+h\,e_k) - a_{i,j}(x)D_j u(x)}{h}\\ &= a_{i,j}(x+h\,e_k)(\Delta_k^h D_j u)(x) + (\Delta_k^h a_{i,j})(x) D_j u(x).\end{aligned}$$

By assumption, $a_{i,j}(x)$ is Lipschitz, therefore

$$|\Delta_k^h a_{i,j}(x)| \;=\; \frac{|a_{i,j}(x+h\,e_k) - a_{i,j}(x)|}{h} \leqslant \frac{\mathrm{Lip}(a_{i,j})\cdot|h|}{|h|} = \mathrm{Lip}(a_{i,j}),$$

where

$$\alpha := \mathrm{Lip}(a_{i,j}) = \sup_{x,y\in\Omega}\frac{|a_{i,j}(x) - a_{i,j}(y)|}{|x-y|}.$$

We may rewrite (4.4) as

$$\begin{aligned}\int_\Omega (Dv^T A(x+h\,e_k) D\Delta_k^h u\mathrm{d}x &= -\int_\Omega (\tilde{g}\Delta_k^h v + \alpha D v)\mathrm{d}x\\ &\leqslant \|g\|_{L^2}\left\|\Delta_k^h v\right\|_{L^2} + \|\alpha\|_{L^2}\|Dv\|_{L^2}\\ &\leqslant (\|\tilde{g}\|_{L^2} + \|\alpha\|_{L^2})\|Dv\|_{L^2}\\ &\leqslant C(K,n)\Big(\|u\|_{W^{1,2}(\Omega)} + \|g\|_{L^2(\Omega)}\Big)\|Dv\|_{L^2}.\end{aligned}$$

This holds for all $v\in C_c^1(\Omega)$ and by density for all $v\in W_0^{1,2}(\Omega)$. So we may choose

$$v = \eta\Delta_k^h u,$$

where $\eta\in C_c^1(\Omega)$ and

$$\mathrm{dist}(\mathrm{supp}(\eta),\partial\Omega) > \frac{d'}{2}.$$

By strict ellipticity (E_1), we have

$$\xi^T A\xi \geqslant \lambda|\xi|^2 \quad \text{for all } \xi\in\mathbb{R}^n, x\in\Omega.$$

If $\eta\geqslant 0$, we have

$$\eta(\Delta_k^h Du)^T A(x+h\,e_k)(\Delta_k^h Du) \geqslant \lambda\eta|\Delta_k^h Du|^2.$$

Therefore, $v=\eta\Delta_k^h u$ in the estimate of rewritten (4.4)

$$\begin{aligned}\lambda\int_\Omega \eta|\Delta_k^h Du|^2\mathrm{d}x &\overset{(E_1)}{\leqslant} \int_\Omega \eta(\Delta_k^h Du)^T A\Delta_k^h Du\\ &\overset{\text{product rule}}{=} \int_\Omega Dv^T A\Delta_k^h Du - \int_\Omega (v\,D\eta)^T A\,\Delta_k^h Du\\ &\leqslant C\big(\|u\|_{W^{1,2}} + \|g\|_{L^2}\big)\|Dv\| - (\downarrow)???.\end{aligned}$$

$$Dv = D(\eta\Delta_k^h u) = D\eta\Delta_k^h u + \eta D\Delta_k^h u.$$

Observe that we may choose $\eta = 1$ on Ω' and $\eta \in C_c^1(\Omega')$ such that $\|D\eta\|_{L^\infty} \leqslant C(n)/d'$. Estimate RHS using this to find

$$\lambda \int_\Omega |D\Delta_k^h u|^2 \mathrm{d}x \leqslant \lambda \int_\Omega \eta |D\Delta_k^h u|^2 \mathrm{d}x \leqslant C\Big(\|u\|_{W^{1,2}(\Omega)} + \|g\|_{L^2(\Omega)} \Big). \qquad \square$$

Theorem 4.20. (Ladyzhenskaya & Uraltseva) *Assume (E_1) and (E_2). Assume $f \in L^q(\Omega)$, $g \in L^{q/2}$ for some $q > n$. Then if u is a $W^{1,2}$ subsolution with $u \leqslant 0$ on $\partial\Omega$, we have*

$$\sup_\Omega u \leqslant C\Big(\|u^+\|_{L^2(\Omega)} + k \Big),$$

where

$$k = \frac{1}{\lambda}\Big(\|f\|_{L^q} + \|g\|_{L^{q/2}} \Big) \quad \textit{and} \quad C = (n, \nu, q, |\Omega|).$$

Proof. (Moser) To expose the main idea, assume that

$$f = 0, g = 0 \quad \Rightarrow \quad k = 0$$

and $c = 0$, $d = 0$. We need to show

$$\sup_\Omega u \leqslant C \|u^+\|_{L^2}.$$

Recall that (1) $u \leqslant 0$ on $\partial\Omega$ means that

$$u^+ = \max\{u, 0\} \in W_0^{1,2}(\Omega).$$

(2) u is a subsolution if

$$B[u, v] \leqslant F(v)$$

for $v \in W_0^{1,2}(\Omega)$ and $v \geqslant 0$, which means that

$$\int_\Omega Dv^T (A\,Du + bu) \mathrm{d}x \leqslant 0$$

for $v \in W_0^{1,2}(\Omega)$ and $v \geqslant 0$.

Main idea: Choose *nonlinear* test functions of the form $v = (u^+)^\beta$ for some $\beta \geqslant 1$. Let $w := u^+$ for brevity. We know that $w \in W_0^{1,2}(\Omega)$. Let

$$H(z) = \begin{cases} z^\beta & 0 \leqslant z \leqslant N, \\ \text{linear} & z > N, \end{cases}$$

i.e.

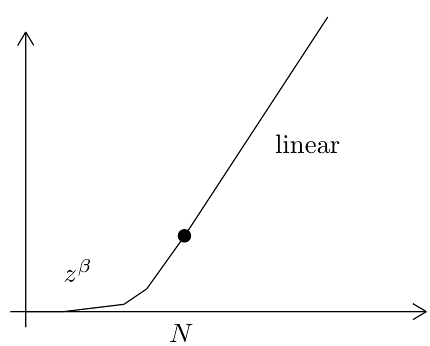

Figure 4.1.

Let

$$v(x) = \int_0^{W(x)} |H'(z)|^2 \mathrm{d}z.$$

Then

$$Dv(x) = |H'(w)|^2 Dw(x). \tag{4.5}$$

Note that $v \geqslant 0$ by construction. Moreover, $|H'(w)|^2 \in L^\infty$ and $w \in W_0^{1,2}(\Omega) \Rightarrow v \in W_0^{1,2}(\Omega)$. We have from (4.5) that

$$\begin{array}{rcl} \displaystyle\int_\Omega Dv^T A\, Du \mathrm{d}x & \leqslant & \displaystyle -\int_\Omega (Dv^T b) u(x) \mathrm{d}x \\ \| & & \\ \displaystyle\int_\Omega |H'(w)|^2 Dw^T A\, Du\, \mathrm{d}x & = & \displaystyle\int_\Omega |H'(w)|^2 Dw^T A\, Dw\, \mathrm{d}x \\ & \geqslant & \displaystyle\lambda \int_\Omega |H'(w)|^2 |Dw|^2 \mathrm{d}x. \end{array}$$

On the other hand,

$$\begin{array}{rcl} \displaystyle\left| -\int_\Omega (Dv^T b) u(x) \mathrm{d}x \right| & = & \displaystyle\left| \int_\Omega |H'(w)|^2 Dw^T b\, u\, \mathrm{d}x \right| \\ & \overset{w=u^+}{=} & \displaystyle\left| \int_\Omega |H'(w)|^2 Dw^T b\, w \mathrm{d}x \right| \\ & \overset{\text{CS}}{\leqslant} & \displaystyle\left(\int_\Omega \underbrace{|H'(w)|^2 |Dw|^2}_{|DH(w)|^2} \mathrm{d}x \right)^{1/2} \left(\int_\Omega |H'(w)|^2 |b|^2 |w|^2 \mathrm{d}x \right)^{1/2}. \end{array}$$

Thus we have

$$\begin{array}{rcl} \displaystyle\lambda \int_\Omega |DH(w)|^2 \mathrm{d}x & \leqslant & \displaystyle\left(\int_\Omega |DH(w)|^2 \mathrm{d}x \right)^{1/2} \left(\int_\Omega |H'(w)|^2 |b|^2 |w|^2 \mathrm{d}x \right)^{1/2} \\ & \overset{\text{AM-GM}}{\leqslant} & \displaystyle\frac{1}{2} \left[\lambda \int_\Omega |DH(w)|^2 \mathrm{d}x + \frac{\|b\|_\infty}{\lambda} \int_\Omega |H'(w)|^2 |w|^2 \mathrm{d}x. \right. \end{array}$$

Therefore

$$\int_\Omega |DH(w)|^2 \mathrm{d}x \leqslant \frac{\|b\|_\infty}{\lambda^2} \int_\Omega |H'(w)|^2 |w|^2 \mathrm{d}x \overset{(E_2)}{\leqslant} \nu^2 \int_\Omega |H'(w)|^2 |w|^2 \mathrm{d}x.$$

By Sobolev's Inequality

$$\|H(w)\|_{L^{2^*}(\Omega)} \leqslant C(n) \|DH(w)\|_{L^2(\Omega)} \leqslant \nu C(n) \|H'(w) w\|_{L^2(\Omega)}.$$

This inequality is independent of N, so take $N \uparrow \infty$. Then $H(w) = w^\beta$, $H'(\omega) = \beta w^{\beta-1}$, so

$$w H'(w) = \beta \omega^\beta.$$

Then

$$\left(\int_\Omega |w|^{\beta 2^*} \mathrm{d}x \right)^{1/2^*} \leqslant \nu C(n) \beta \left(\int_\Omega |w|^{2\beta} \mathrm{d}x \right)^{1/2}.$$

Thus we have

$$\|w\|_{2^*\beta} \leqslant (\nu C(n) \beta)^{1/\beta} \|w\|_{2\beta}, \quad \beta \geqslant 1. \tag{4.6}$$

Note that $2^* = 2n/(n-2) > 2$. Let $r := n/(n-2)$. Then iterate (4.6):

$$\begin{array}{lll} \beta = 1 & \Rightarrow & \|w\|_{2r} \leqslant (\nu C(n)) \|w\|_2 \\ \beta = r & \Rightarrow & \|w\|_{2r^2} \leqslant (\nu C(n) r)^{1/r} \|w\|_{2r} \leqslant (\nu C(n) r)^{1/r} (\nu C(n)) \|w\|_2. \end{array}$$

By induction,

$$\begin{array}{rcl} \|w\|_{2r^{m+1}} & \leqslant & (\nu C(n))^{1 + \frac{1}{r} + \cdots + \frac{1}{r^m}} (r)^{\frac{1}{r} + \frac{2}{r^2} + \frac{m}{r^m}} \|w\|_2 \\ & \leqslant & (\nu C(n))^{\frac{1}{1 - 1/r}} (r)^{1/(1-1/r)^2} \|w\|_2. \end{array}$$

Let $m \to \infty$ and obtain

$$\|w\|_{L^\infty} = \sup u^+ \leqslant C \|u^+\|_2. \qquad \square$$

4.5 The Weak Harnack Inequality

Label two common assumptions for this section

(1). Assume (E_1), (E_2).

(2). Also assume $f\in L^q(\Omega)$, $g\in L^{q/2}(\Omega)$ for some $q>n$.

Theorem 4.21. (Local boundedness) *Assume (1), (2). Assume u is a subsolution. Then for any ball $B(y,2R)\subset\Omega$ and $p>1$*

$$\sup_{B(y,R)} u\leqslant C\Big(R^{-n/p}\|u^+\|_{L^p(B(y,2R))}+k(R)\Big),$$

where

$$k(R)=\frac{R^{1-n/q}}{\lambda}\Big(\|f\|_q+R^{1-n/q}\|g\|_{q/2}\Big)$$

and

$$C=C\bigg(n,\frac{\Lambda}{\lambda},|\Omega|,\nu\bigg).$$

Theorem 4.22. (Weak Harnack Inequality) *Assume (1), (2). If u is a $W^{1,2}(\Omega)$* supersolution *and $u\geqslant 0$ in a ball $B(y,4R)\subset\Omega$, then*

$$R^{-n/p}\|u\|_{L^p(B(y,2R))}\leqslant C\bigg(\inf_{B(y,R)} u+k(R)\bigg)$$

for every $1\leqslant p<n/(n-2)$ with C and k as before.

Now, let us consider the consequences of Theorem 1 and 2.

Theorem 4.23. (Strong Harnack Inequality) *Assume (1), (2). Assume u is a $W^{1,2}$ solution with $u\geqslant 0$. Then*

$$\sup_{B(y,R)} u\leqslant C\bigg(\inf_{B(y,R)} u+k(R)\bigg).$$

Theorem 4.24. (Strong Maximum Principle) *Assume (1), (2) and (E_3). Assume Ω connected. Suppose u is a $W^{1,2}$ subsolution. If for some ball $B(y,R)\subsetneq\Omega$, we have*

$$\sup_B u=\sup_\Omega u,$$

then $u=\mathrm{const}$.

Proof. Suppose $M=\sup_\Omega u$. Also suppose $B(y,4R)\subsetneq\Omega$ and $\sup_{B(y,4R)} u=M$. Let $v=M-u$, then $Lv=-Lu\leqslant 0$ (i.e. supersolution) and $v\geqslant 0$. Apply weak Harnack inequality with $p=1$:

$$R^{-n}\int_{B(y,2R)}(M-u)\mathrm{d}x\leqslant C\bigg(\inf_{B(y,R)}(M-u)\bigg)=0.$$

$\Rightarrow\{u=M\}$ is open. Even though u is not continuous, it is still true that $\{u=M\}$ is relatively closed in Ω. Then $\{u=M\}=\Omega$ since Ω is connected. □

Theorem 4.25. (DeGiorgi, Nash) *Assume (1), (2). Assume $u\in W^{1,2}$ solves $Lu=g+\operatorname{div} f$. Then u is locally Hölder continuous and for any ball $B_0=B(y,R_0)\subset\Omega$ and $0<R\leqslant R_0$. Then*

$$\operatorname{osc}_{B(y,R)}u\leqslant C\,R^\alpha\bigg(R_0^{-\alpha}\sup_{B_0}|u|+k\bigg).$$

Here, C and k are as before and $\alpha=a(n,\Lambda/\lambda,\nu,R,q)$.

Proof. To avoid complications work with the simpler setting

$$Lu=\operatorname{div}(A\,Du)=0,$$

i.e. $b = c = f = 0$, $d = g = 0$. Assume without loss $R \leqslant R_0/4$. Let

$$\begin{aligned} M_0 &:= \sup_{B_0} |u|, \\ M_1 &:= \sup_{B_R} u, \quad m_1 := \inf_{B_R} u, \\ M_4 &:= \sup_{B_{4R}} u, \quad m_4 := \inf_{B_{4R}} u. \end{aligned}$$

Let $\omega(R) := \operatorname{osc}_{B_R} u = M_1 - m_1$. Observe that $M_4 - u \geqslant 0$ on B_{4R} and $L(M_4 - u) = 0$. Similarly, $u - m_4 \geqslant 0$ on B_{4R} and $L(u - m_4) = 0$. Thus, we can apply the weak Harnack inequality with $p = 1$ to obtain

$$R^{-n} \int_{B_{2R}} (M_4 - u) \mathrm{d}x \leqslant C\left(\inf_{B_R} (M_4 - u)\right) = C(M_4 - M_1).$$

Likewise,

$$R^{-n} \int_{B_{2R}} (u - m_4) \mathrm{d}x \leqslant C\left(\inf_{B_R} (u - m_4)\right) = C(m_1 - m_4).$$

Add both inequalities to obtain

$$\frac{1}{R^n} \int_{B_{2R}} (M_4 - m_4) \mathrm{d}x = C_n (M_4 - m_4) \leqslant C\left[\underbrace{(M_4 - m_4)}_{\operatorname{osc}_{B_{4R}} u} - \underbrace{(M_1 - m_1)}_{\operatorname{osc}_{B_R} u}\right].$$

Rewrite as

$$\omega(R) \leqslant \gamma \omega(4R)$$

for some $\gamma > 1$. Fix $r \leqslant R_0$. Choose m such that

$$\frac{1}{4^m} R_0 \leqslant r < \frac{1}{4^{m-1}} R_0.$$

Observe that $\omega(R)$ is non-decreasing since $\omega(r) = \sup_{B_r} u - \inf_{B_r} u$. Therefore

$$\begin{aligned} \omega(r) &\leqslant \omega\left(\frac{1}{4^{m-1}} R_0\right) \\ &\leqslant \gamma^{m-1} \omega(R_0). \\ &\leqslant \left(\frac{r}{R_0}\right)^{\log r / \log 4} \omega(R_0), \end{aligned}$$

where we used

$$\frac{1}{4^m} \leqslant \frac{r}{R_0} < \frac{1}{4^{m-1}},$$

therefore

$$\begin{aligned} -m \log 4 \leqslant \log(r/R_0) &< (-m-1) \log 4 \\ \Leftrightarrow m \geqslant -\log(r/R_0)/\log 4 &> (m-1). \end{aligned}$$

□

5 Calculus of Variations

General set-up:

$$I[u] = \int_\Omega F(Du(x)) \mathrm{d}x.$$

Here, we have $u: \Omega \to \mathbb{R}^m$, $m \geqslant 1$. $Du: \Omega \to \mathbb{M}^{m \times n}$. Minimize I over $u \in \mathcal{A}$, where $\mathcal{A}$ is a class of admissible functions.

Example 5.1. *(Dirichlet's principle)* Let Ω be open and bounded and $u: \Omega \to \mathbb{R}$, $g: \Omega \to \mathbb{R}$ given,

$$I[u] = \int_\Omega \left(\frac{1}{2} |Du|^2 - g u\right) \mathrm{d}x$$

and $\mathcal{A}=W^{1,2}(\Omega)$. The terms have the following meanings:

$|Du|^2$. Represents the strain energy in a membrane.

$g\,u$. Is the work done by the applied force.

General principles:

1. Is $\inf_{\mathcal{A}} I[u]>-\infty$?
2. Is $\inf_{\mathcal{A}} I[u]=\min_{\mathcal{A}} I[u]$? (This will be resolved by the *"Direct Method"* due to Hilbert.)

To show 1.): Suppose $g\in L^2(\Omega)$. Then

$$\begin{aligned}\left|\int_\Omega g\,u\,\mathrm{d}x\right| &\leqslant \|g\|_{L^2}\|u\|_{L^2}\\ &\leqslant \frac{1}{2}\left(\varepsilon\|u\|_{L^2}^2+\frac{1}{\varepsilon}\|g\|_{L^2}^2\right).\end{aligned}$$

By the Sobolev Inequality,

$$\|u\|_{L^{2^*}}\leqslant C(n)\|Du\|_{L^2}.$$

Moreover, $2^*>2$ and

$$\begin{aligned}\|u\|_{L^2} &\overset{\text{Hölder's}}{\leqslant} \|u\|_{L^{2^*}}|\Omega|^{1/n}\\ &\leqslant C(n,\Omega)\|Du\|_{L^2}.\end{aligned}$$

Then

$$\begin{aligned}I[u] &= \frac{1}{2}\int_\Omega |Du|^2\mathrm{d}x-\int_\Omega g\,u\,\mathrm{d}x\\ &\geqslant \frac{1}{2}\|Du\|_{L^2}^2-\frac{1}{2}\left(\varepsilon C\|Du\|_{L^2}^2+\frac{1}{\varepsilon}\|g\|_{L^2}^2\right)\\ &\geqslant \frac{1}{4}\|Du\|_{L^2}^2-\frac{1}{2\varepsilon}\|g\|_{L^2}^2\\ &\overset{(*)}{\geqslant} c\|u\|_{W_0^{1,2}}^2-\frac{1}{2\varepsilon}\|g\|_{L^2}^2,\end{aligned}$$

where the step $(*)$ uses the Sobolev inequality again, with a suitable ε chosen.

This is called a *coercivity bound*. In particular,

$$\inf_u I[u]\geqslant-\frac{1}{2\varepsilon}\|g\|_{L^2}^2>-\infty.$$

Since $\inf I[u]>-\infty$, there is some sequence u_k such that $I[u_k]\to\inf I[u_k]$.

Bounds on $\{u_k\}$:

$$\begin{aligned}I[u] &= \frac{1}{2}\int_\Omega |Du|^2\mathrm{d}x-\int_\Omega g\,u\,\mathrm{d}x\\ &\leqslant \frac{1}{2}\left(\int_\Omega |Du|^2+|u|^2\mathrm{d}x\right)+\frac{1}{2}\int_\Omega |g|^2\mathrm{d}x\\ &= \frac{1}{2}\left(\|u\|_{W_0^{1,2}}^2+\|g\|_{L^2}^2\right).\end{aligned}$$

By coercivity, we have

$$\|u_k\|_{W_0^{1,2}(\Omega)}^2 \leqslant \frac{1}{C}\left[\underbrace{I[u_k]}_{*}+\underbrace{\frac{1}{2\varepsilon}\|g\|_{L^2}^2}_{\text{fixed!}}\right],$$

where term $*$ is uniformly bounded because $I[u_k]\to\inf$. We could say $I[u_k]\leqslant\inf+1$.

The main problem is: We cann only assert that there is a *weakly* converging subsequence. That is, $u_{k_j}\rightharpoonup u$ in $W_0^{1,2}(\Omega)$, where we relabel the subsequence u_{k_j} as u_k.

Theorem 5.2. *$I[u]$ is weakly lower semicontinuous. That is, if $v_k \rightharpoonup v$, then*

$$I[v] \leqslant \liminf_{k\to\infty} I[v_k].$$

Assuming the theorem, we see that $I[u]$ is a minimizer. Indeed,

$$I[u] \overset{\text{w.l.s.c.}}{\leqslant} \liminf_{k\to\infty} I[u_k] = \inf_{v\in\mathcal{A}} I[v] \leqslant I[u].$$

Aside: $I[u]$ is also strictly convex $\Rightarrow u$ is a minimizer:

$$I\left[\frac{v_1+v_2}{2}\right] \leqslant \frac{1}{2}(I[v_1]+I[v_2])$$

with equality only if $v_1 = \alpha v_2$ for some $\alpha \in \mathbb{R}$.

Proof. Assume two *distinct* minimizers $u_1 \neq \alpha u_2$. Then

$$I\left[\frac{u_1+u_2}{2}\right] < \frac{1}{2}(I[u_1]+I[u_2]) = \min_{v\in\mathcal{A}} I[v],$$

which contradicts the definition of the minimum. □

Theorem 5.3. *Assume $F: M^{m\times n} \to \mathbb{R}$ is* convex *and $F \geqslant 0$. Then*

$$I[u] = \int_\Omega F(Du(x))\mathrm{d}x$$

is weakly lower semicontinuous in $W_0^{1,p}(\Omega)$ for $1 < p < \infty$.

Proof. From homework, we know that $F(A) = \lim_{N\to\infty} F_N(A)$ where F_N is an increasing sequence of piecewise affine approximations. Since f_N is piecewise affine, if

$$\begin{aligned} u_k &\rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega) \\ Du_k &\rightharpoonup Du \quad \text{in } L^p(\Omega), \end{aligned}$$

we have

$$\int_\Omega F_N(Du_k)\mathrm{d}x \to \int_\Omega F_N(Du)\mathrm{d}x.$$

Thus,

$$\begin{aligned} \int_\Omega F_N(Du)\mathrm{d}x &= \lim_{k\to\infty}\int_\Omega F_N(Du_k)\mathrm{d}x \\ F_N \text{ increasing} \rightarrow \qquad &\leqslant \liminf_{k\to\infty}\int_\Omega F(Du_k)\mathrm{d}x \\ &= \liminf_{k\to\infty} I[u_k]. \end{aligned}$$

Now let $N\to\infty$, and use the monotone convergence theorem to find

$$I[u] = \int_\Omega f(Du)\mathrm{d}x \leqslant \liminf_{k\to\infty} I[u_k].$$ □

Basic issue: Suppose $f(x)$ is as given in this picture:

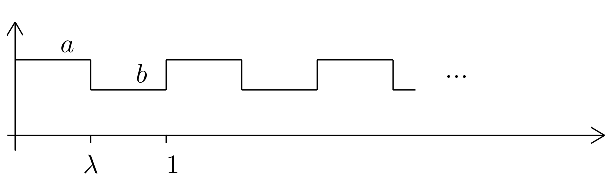

Figure 5.1. $f(x)$.

Consider $g_k(x) = f(k\,x)$, $k \geqslant 1$, $x \in [0,1]$. This just makes f oscillate faster. We then know that

$$g_k \underset{L^\infty}{\overset{*}{\rightharpoonup}} \lambda a + (1-\lambda) b.$$

Suppose F is a nonlinear function. Consider the sequence

$$\begin{aligned} G_k(x) &= F(g_k(x)) \\ &= \begin{cases} F(a) & \text{when } g_k(x) = a, \\ F(b) & \text{when } g_k(x) = b. \end{cases} \end{aligned}$$

Then

$$G_k \rightharpoonup G = \lambda F(a) + (1-\lambda) F(b).$$

But then in general

$$\begin{aligned} G &= \text{weak-}*\lim F(g_k) \neq F(\text{w-}*\lim g_k) \\ &= F(\lambda a + (1-\lambda) b) \end{aligned}$$

However if F is *convex*, we do have an *inequality*

$$F(g) \leqslant \text{w-}*\lim F(g_k).$$

Fix $m = 1$, that is $Du: \Omega \to \mathbb{R}^n$, write $F = F(z)$ for $z \in \mathbb{R}^n$.

Why convexity? Let $v \in W_0^{1,p}(\Omega)$, consider $i(t) = I[u + t\,v]$. If u is a critical point $I \Rightarrow i'(0) = 0$.

$$i'(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega F(Du + t\,Dv) \mathrm{d}x = \int_\Omega DF(Du + t\,Dv) \cdot Dv \,\mathrm{d}x.$$

So,

$$0 = i'(0) = \int_\Omega DF(Du) \cdot Dv \,\mathrm{d}x. \tag{5.1}$$

This is the weak form of the Euler-Lagrange equations

$$\begin{aligned} 0 &= -\operatorname{div}(DF(Du(x))) \quad \text{in}\,\Omega, \\ u &= g \quad \text{on}\,\partial\Omega. \end{aligned}$$

With index notation

$$i'(t) = \int_\Omega \frac{\partial F}{\partial z_j}(Du + t\,Dv) \cdot \frac{\partial v}{\partial x_j} \,\mathrm{d}x.$$

If u is a minimum, $i''(0) \geqslant 0$.

$$i''(t) = \int_\Omega \frac{\partial^2 F}{\partial z_j \partial z_k}(Du + t\,Dv) \cdot \frac{\partial v}{\partial x_j} \frac{\partial v}{\partial x_k} \,\mathrm{d}x$$

Thus,

$$0 \leqslant \int_\Omega \frac{\partial^2 F}{\partial z_j \partial z_k}(Du) \cdot \frac{\partial v}{\partial x_j} \frac{\partial v}{\partial x_k} \,\mathrm{d}x = \int_\Omega Dv^T D^2 F(Du)\, Dv \,\mathrm{d}x. \tag{5.2}$$

A useful family of test functions: Consider

$$\rho(s) = \begin{cases} \vdots & \vdots \\ s & 0 \leqslant s < 1 \\ 2 - s & 1 \leqslant s < 2 \\ \vdots & \vdots \\ \text{extended} & \text{periodically} \end{cases}$$

Fix $\xi \in \mathbb{R}^n$ and $\zeta \in C_c^\infty(\Omega)$. Consider

$$v_\varepsilon(x) = \varepsilon \zeta(x) \underbrace{\rho\!\left(\frac{x \cdot \xi}{\varepsilon}\right)}_{(*)},$$

where the term $(*)$ oscillates rapidly in the direction ξ.

$$\frac{\partial v_\varepsilon}{\partial x_j} = \underbrace{\varepsilon\frac{\partial \zeta}{\partial x_j}\rho\left(\frac{x\cdot\xi}{\varepsilon}\right)}_{O(\varepsilon)} + \underbrace{\zeta(x)\rho'\left(\frac{x\cdot\xi}{\varepsilon}\right)\xi_j}_{O(1)}.$$

Therefore,

$$\frac{\partial v_\varepsilon}{\partial x_j}\frac{\partial v_\varepsilon}{\partial x_k} = \zeta(x)^2\left(\rho'\left(\frac{x\cdot\xi}{\varepsilon}\right)\right)^2\xi_j\xi_k + O(\varepsilon) = \zeta^2\xi_j\xi_k + O(\varepsilon).$$

Substitute in (5.2) and pass to limit

$$0 \leqslant \int_\Omega \zeta^2(x)\left[\xi_k\frac{\partial^2 F}{\partial z_j \partial z_k}(Du)\xi_j\right]\mathrm{d}x.$$

Since ζ is arbitrary, we have

$$\xi^T D^2F(Du)\xi \geqslant 0, \quad \xi\in\mathbb{R}^n.$$

So, F is convex $\Rightarrow$ (5.1) is an elliptic PDE.

Theorem 5.4. *Assume $m=1$. Then I is w.l.s.c. $\Leftrightarrow F$ is convex in $W^{1,p}(\Omega)$ for $1<p<\infty$.*

Proof. Fix $z\in\mathbb{R}^n$ and suppose $\Omega = Q = [0,1]^n$. Let $u = z\cdot x$. *Claim:* For every $v\in C_c^\infty(\Omega)$, we have

$$I[u] = \int_\Omega F(z)\,\mathrm{d}x = F(z) \leqslant \int_\Omega F(z+Dv)\mathrm{d}x.$$

This is all we have to prove, because we may choose smooth functions to find $\xi^T D^2 F(z)\xi\geqslant 0$. For every k divide Q into subcubes of side length $1/2^k$. Let x_l denote the center of cube Q_l, where $1\leqslant l\leqslant 2^{nk}$.

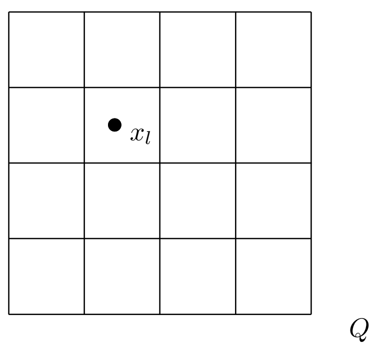

Figure 5.2.

Define a function u_k as follows:

$$u_k(x) = \frac{1}{2^k}v(2^k(x-x_l)) + u(x)$$

for x in Q_l.

$$Du_k(x) = Dv(2^k(x-x_l)) + z$$

for x in Q_l. Thus, $Du_k \rightharpoonup Du = z$.

Since $I[u]\leqslant \liminf_{k\to\infty} I[u_k]$, we have

$$\begin{aligned} F(z) &\leqslant \liminf_{k\to\infty}\sum_{l=1}^{2^{nk}}\int_{Q_l} F(z+Dv(2^k(x-x_l)))\,\mathrm{d}x \\ &= \liminf_{k\to\infty} 2^{nk}\int_{Q_l} F(z+Dv(2^k(x-x_l)))\,\mathrm{d}x \quad \text{(integral same in every cube)} \\ &= \int_\Omega F(z+Dv)\mathrm{d}x. \end{aligned}$$

$\square$

Problem in higher dimensions: Typical example: $u: \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$.

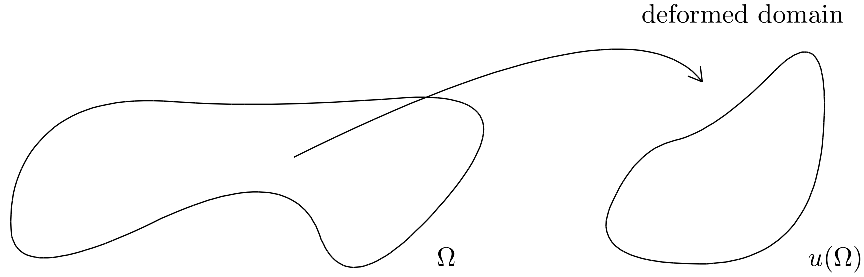

Figure 5.3.

Typically,

$$F(Du) = \underbrace{\frac{1}{2} Du^T Du}_{\text{convex}} + \underbrace{(\det(Du))^p}_{\text{not convex}}.$$

5.1 Quasiconvexity

(cf. Ch. 3, little Evans) $u: \Omega \to \mathbb{R}^m$, $m \geqslant 2$

$$\mathcal{A} = \left\{ u \in W^{1,p}(\Omega, \mathbb{R}^m) : u = g \text{ on } \partial\Omega \right\}$$

$1 < p < \infty$, Ω open, bounded,

$$I[u] = \int_\Omega F(Du(x))\,\mathrm{d}x$$

with $F: \mathbb{M}^{m \times n} \to \mathbb{R}$, C^∞. Always assume F coercive, that is

$$F(A) \geqslant c_1 |A|^p - c_2.$$

$\Rightarrow$ The main issue is the weak lower semicontinuity of I.

Question: What ‘structural assumptions’ must F satsisfy? if $m = 1$, we know that F should be *convex*. This is sufficient for all n. Is this necessary?

Convexity is bad because it contradicts material frame indifference.

Rank-one convexity: Let’s replicate a calculation already done: Let $i(t) := I[u + t\ v]$, $t \in [-1, 1]$. Assume $i'(0) = 0$, $i''(0) \geqslant 0$.

$$i(t) = \int_\Omega F(Du + t\,Dv)\,\mathrm{d}x.$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t} F(Du + t\,Dv)\mathrm{d}x = \int_\Omega \frac{\partial F}{\partial A_{i,k}}(Du + t\,Dv)\frac{\partial v_i}{\partial x_k}\mathrm{d}x$$

(Use summation convention.)

$$0 = i'(0) \Rightarrow 0 = \int_\Omega \frac{\partial F}{\partial A_{i,k}}(Du)\frac{\partial v_i}{\partial x_k}\mathrm{d}x.$$

This is the weak form of the Euler-Lagrange equations

$$-\frac{\partial}{\partial x_k}\left(\frac{\partial F}{\partial A_{i,k}}(Du)\right) = 0 \tag{5.3}$$

for $i = 1, \ldots, m$, so we have a *system*. Now consider $i''(0) \geqslant 0$.

$$i''(0) = \int_\Omega \frac{\partial^2 F}{\partial A_{i,k}\partial A_{j,l}}(Du)\frac{\partial v_i}{\partial x_k}\frac{\partial v_j}{\partial x_l}\mathrm{d}x \geqslant 0. \tag{5.4}$$

As before, consider oscillatory test functions:

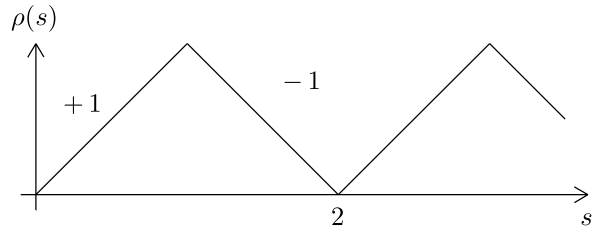

Figure 5.4.

Fix $\eta \in \mathbb{R}^m$, $\xi \in \mathbb{R}^n$, $\zeta \in C_c^{\infty}(\Omega; \mathbb{R})$.

$$v(x) = \varepsilon \zeta(x) \rho\left(\frac{x \cdot \xi}{\varepsilon}\right) \eta.$$

Then

$$\frac{\partial v_i}{\partial x_k} = \varepsilon \zeta'(x) \rho\left(\frac{x \cdot \xi}{\varepsilon}\right) \eta + \zeta(x) \rho'\left(\frac{x \cdot \xi}{\varepsilon}\right) \eta_i \xi_k.$$

Thus

$$\frac{\partial v_i}{\partial x_k} \frac{\partial v_j}{\partial x_l} = \zeta(x)^2 \eta_i \eta_j \xi_k \xi_l + O(\varepsilon)$$

Substitute in (5.4) and let $\varepsilon \to 0$,

$$0 \leqslant \int_{\Omega} \underbrace{\zeta^2(x)}_{\text{arbitrary}} \left[\frac{\partial^2 F}{\partial A_{i,k} \partial A_{j,l}}\right] \eta_i \eta_j \xi_k \xi_l \, \mathrm{d}x.$$

This suggests that F should satisfy

$$(\eta \otimes \xi)^T D^2 F (\eta \otimes \xi) \geqslant 0 \tag{5.5}$$

for every $\eta \in \mathbb{R}^m$, $\xi \in \mathbb{R}^n$. $\eta \otimes \xi = \eta \xi^T$ is a rank-one matrix.

Note: F is convex if $B^T D^2 F(A) B \geqslant 0$ for every $B \in \mathbb{M}^{m \times n}$. However, we only need B to be rank one in (5.5). (5.5) is known as the Legendre-Hadamard condition. It ensures the *ellipticity* of the *system* (5.3). Thus, we see that if I is w.l.s.c. then F should be rank-one convex. Q: Is that sufficient?

Definition 5.5. (Morrey, 1952) *F is* quasiconvex (QC) *if*

$$F(A) \leqslant \int_Q F(A + Dv(x)) \, \mathrm{d}x$$

for every $A \in \mathbb{M}^{m \times n}$ and $v \in C_c^{\infty}(Q, \mathbb{R}^m)$. Here Q is the unit cube in $\mathbb{R}^n$.

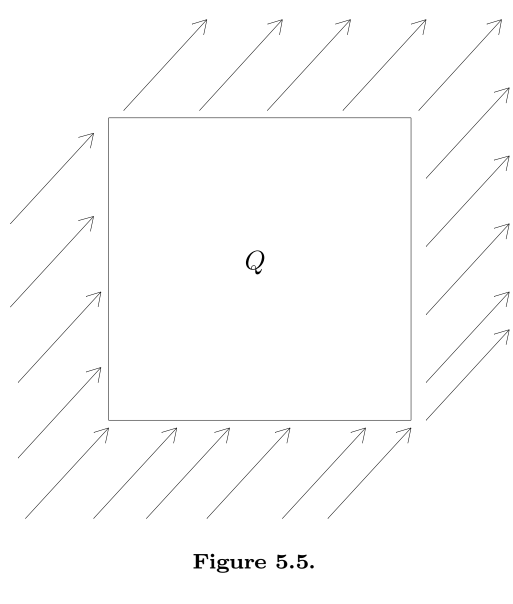

Figure 5.5.

Subject the boundary of a cube to an affine deformation $A(x)$. Then $u = A x$ for $x \in Q$ satisfies the boundary condition $Du(x) = A$ for $x \in \partial Q$.

$$I[u] = \int_Q F(Du)\,\mathrm{d}x = F(A).$$

Thus (QC) implies $I[u] \leqslant I[u+v]$ for any $v \in C_c^\infty(Q) \Rightarrow$ affine deformation is the best.

Examples of QC functions:

1. $F(A) = \det(A)$ or a minor of A

Definition 5.6. (Ball) *F is* polyconvex (PC) *if F is a convex function of the minors of A.*

What's known:

Theorem 5.7. (Morrey) *Assume $F \in C^\infty$ satisfies the growth condition*

$$|F(A)| \leqslant C(1 + |A|^p) \tag{5.6}$$

with some $C > 0$. Then I is w.l.s.c. $\Leftrightarrow F$ is QC.

Remark 5.8.

$$\text{Convex} \overset{\not\Leftarrow}{\Rightarrow} \text{Polyconvex} \overset{\not\Leftarrow}{\Rightarrow} \text{Quasiconvex} \overset{\not\Leftarrow(*)}{\Rightarrow} \text{Rank-one-convex (RC)}.$$

$(*)$ is known for $m \geqslant 3$, $n \geqslant 2$ (Světrak, '92), but not known for $m = 2$, $n \geqslant 2$.

We'll prove that if $u_k \in W^{1,p}$ for $p > n$ and $u_k \rightharpoonup u \Rightarrow \det(Du_k) \rightharpoonup \det(Du)$ in $L^{p/n}$. (*compensated compactness in $L^{p/n}$*)

If $A_k(x) \in L^{p/n}(\Omega, \mathbb{M}^{m\times n})$ and $A_k \rightharpoonup A$, it is not true that $\det(A_k) \rightharpoonup \det(A)$.

Note 5.9. "$\Rightarrow$" is straightforward. Simply choos $u(x) = A x$ and $u_k = A x + v_k(x)$ ($v_k \leftarrow$ periodic scaling).

Assume F is QC and statisfies (5.6).

Lemma 5.10. *There is a $C > 0$ such that*

$$|DF(A)| \leqslant C(1 + |A|^{p-1}).$$

Proof. Fix $A \in \mathbb{M}^{m\times n}$ and a rank-one matrix $\eta \otimes \xi$ with η, ξ coordinate vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$. We know that QC $\Rightarrow$ RC, therefore the function

$$f(t) = F(A + t(\eta \otimes \xi))$$

is convex. By homework, we know that $f(t)$ is locally Lipschitz and

$$|DF(A)(\eta \otimes \xi)| = |f'(0)| \leqslant \frac{C}{r} \max_{t \in [-r,r]} |f(t)|.$$

Then

$$\begin{aligned} |f(t)| &= |F(A + t(\eta \otimes \xi))| \\ &\overset{(5.6)}{\leqslant} C(1 + |A|^p + t^p |\eta \otimes \xi|^p) \\ &\leqslant C(1 + |A|^p + r^p). \end{aligned}$$

Choose $r = \max(1, |A|)$ to find

$$|f'(0)| \leqslant C(1 + |A|^{p-1}).$$

□

Proof. (of Theorem 5.7) Assume F is QC, show I is w.l.s.c.

QC tells you...

$$⨍_Q F(D(A x))\mathrm{d}x = F(A) \leqslant ⨍_Q F(A + Dv(x)$$

For w.l.s.c., we want to show... If $u_k \rightharpoonup u$ in $W^{1,p}$, then

$$\int_\Omega F(Du)\mathrm{d}x \leqslant \liminf_{k\to\infty} \int_\Omega F(Du_k)\mathrm{d}x.$$

Idea: Subdivide domain Ω into small cubes:

$$\int_\Omega F(Du)\mathrm{d}x \approx \int_\Omega F(\text{affine approximation to } Du)\mathrm{d}x \overset{\text{QC}}{\leqslant} \int_\Omega F(Du_k)\mathrm{d}x + \text{errors}.$$

1) Assume $u_k \rightharpoonup u$ in $W^{1,p}(\Omega, \mathbb{R}^m)$. Then

$$\sup_k \|Du_k\|_{L^p(\Omega, \mathbb{M}^{m\times n})} < \infty$$

by the uniform boundedness principle (Banach-Steinhaus). By considering a subsequence, we have

$$u_k \to u \text{ in } L^p(\Omega, \mathbb{R}^m)$$

(cf. Lieb&Loss) Define the measures

$$\mu_k(\mathrm{d}x) = (1 + |Du_k|^p + |Du|^p)\mathrm{d}x.$$

By the uniform bounds,

$$\sup_k \mu_k(\Omega) < \infty.$$

Then there is a subsequence $\mu_k \rightharpoonup \mu$ with

$$\underbrace{\mu(\Omega)}_{\text{concentration measure}} \leqslant \liminf_{k\to\infty} \mu_k(\Omega).$$

Suppose H is a hyperplane perpendicular to the unit vector e_k. Therefore, $\mu(\Omega \cap H) \neq 0$ for at most countably many hyperplanes.

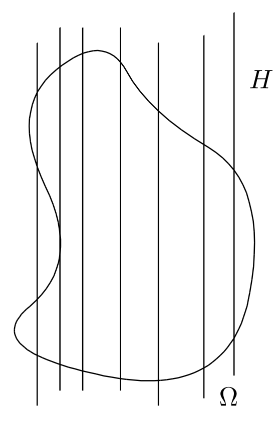

Figure 5.6.

By translating the axes if necessary, we can assert that if $\mathbb{Q}_i$ denotes the dyadic lattice with side length 2^{-i}, then $\mu(\partial Q_l) = 0$ for every $Q_l \in \mathbb{Q}_i$ and every i. Let $(Du)_i$ denote the piecewise constant function with value

$$⨍_{Q_l} Du(x)\mathrm{d}x$$

on the cube Q_l. By Lebesgue's Differentiation Theorem, $(Du)_i \to Du$ a.e. for $i \uparrow \infty$ in $L^p(\Omega, \mathbb{M}^{m\times n})$. Then

$$\int_\Omega |F((Du)_i) - F(Du)|\mathrm{d}x \to 0$$

by DCT.

2) Fix $\varepsilon > 0$, choose $\Omega' \subset\subset \Omega$ such that

$$\int_{\Omega \backslash \Omega'} F(Du) \mathrm{d}x < \varepsilon.$$

Choose i so large that

$$\begin{aligned} \|Du - (Du)_i\|_{L^p} &< \varepsilon, \\ \|F(Du) - F((Du)_i)\|_{L^1} &< \varepsilon. \end{aligned}$$

Aside: Preview: Where is this proof going?

$$I[u_k] \geqslant$$

$$\begin{aligned} I[u_k] &\geqslant \sum_{l=1}^{m} \int_{Q_l} F(Du_k) \mathrm{d}x \\ &= \sum_{l=1}^{m} \int_{Q_l} F(Du + (Du_k - Du)) \mathrm{d}x \\ &\geqslant \sum_{l=1}^{m} \int_{Q_l} F(Du) \mathrm{d}x + E_1 \\ &\geqslant \sum_{l=1}^{m} \int_{Q_l} F(\underbrace{(Du)_i}_{\text{piecewise affine}}) \mathrm{d}x + E_1 + E_2 \\ &\overset{\text{QC}}{\geqslant} I[u] + E_1 + E_2 + E_3. \end{aligned}$$

End aside. (Let's not complete this proof.) □

5.2 Null Lagrangians, Determinants

$$I[u] = \int_{\Omega} F(Du) \mathrm{d}x$$

for $u: \Omega \to \mathbb{R}^m$, $F: \mathbb{M}^{m \times n} \to \mathbb{R}$. The Euler-Lagrange equations read

$$\frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial A_{i,j}}(Du)\right) = 0, \quad i = 1, \ldots, m. \tag{5.7}$$

Definition 5.11. *F is a* null-Lagrangian *if (5.7) holds for every $u \in C^2(\Omega)$.*

$u: \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$

Theorem 5.12. det *is a null-Lagrangian. The associated Euler-Lagrange equation is*

$$\frac{\partial}{\partial x_j}(\operatorname{cof}(Du)_{i,j}) = 0, \quad i = 1, \ldots, n. \tag{5.8}$$

Proof. *Claims:*

1. A matrix identity:

$$\frac{\partial(\det A)}{\partial A_{l,m}} = (\operatorname{cof} A)_{l,m}$$

2. If $A = Du$, then (5.8) holds.

$$(\operatorname{cof} A)_{l,m} = (n-1) \times (n-1) \det(A \text{ without row } l, \text{ column } m).$$

Algebra identity:

$$A^{-1} = \frac{1}{\det A}(\operatorname{cof} A)^T.$$

$$(\det A)\mathrm{Id} = A^T(\operatorname{cof} A).$$

Let B denote cof A.

$$\det A\,\delta_{i,j} = A_{k,i} B_{k,j} \tag{5.9}$$

Claim 1 follows from (5.9), since $(\operatorname{cof} A)_{l,m}$ depends only on $A_{i,j}$ $i \neq l, j \neq m$.

Differentiate both sides w.r.t. x_j:

$$\begin{aligned}
\text{LHS:} \qquad & \frac{\partial}{\partial x_j}(\det A)\delta_{i,j} \\
= \quad & \frac{\partial}{\partial x_j}(\det A) \\
= \quad & \frac{\partial(\det A)}{\partial A_{l,m}} \cdot \frac{\partial A_{l,m}}{\partial x_i} \\
\overset{\text{Claim 1}}{=} \quad & \boxed{B_{l,m}\frac{\partial A_{l,m}}{\partial x_i}},
\end{aligned}$$

where we have used summation over repeated indices.

$$\text{RHS:} \quad \boxed{\frac{\partial A_{k,i}}{\partial x_j} B_{k,j}} + A_{k,i} \underbrace{\frac{\partial B_{k,j}}{\partial x_j}}_{\text{want to say this is 0.}}$$

$\square$ terms are typically not the same for arbitrary matrices $A(x)$. However, if $A(x) = Du(x)$, then

$$B_{k,j}\frac{\partial A_{k,i}}{\partial x_j} = B_{k,j}\frac{\partial^2 u_k}{\partial x_i \partial x_j} = B_{l,m}\frac{\partial^2 u_l}{\partial x_i \partial x_m} = B_{l,m}\frac{\partial A_{l,m}}{\partial x_i}$$

Comparing terms, we have

$$A_{k,i}\frac{\partial B_{k,j}}{\partial x_j} = 0, \quad i = 1, \ldots, n$$

or $(Du)^T \operatorname{div}(\operatorname{cof} Du) = 0 \in \mathbb{R}^n$.

$$\operatorname{cof} Du = n \times n\,\text{matrix}\begin{pmatrix} — \\ — \\ — \end{pmatrix}$$

$$\operatorname{div}(\operatorname{cof} Du) = (\ \updownarrow\ )$$

If Du is invertible, we have div(cof Du) = 0 as desired. If not, let $u_\varepsilon = u + \varepsilon x$. Then $Du_\varepsilon = Du + \varepsilon I$ is invertible for arbitrarily small $\varepsilon > 0$ and

$$\operatorname{div}(\operatorname{cof}(Du_\varepsilon)) = 0.$$

Now let $\varepsilon \searrow 0$. $\square$

Theorem 5.13. (Morrey, Reshetnyak) (Weak continuity of determinant) *Suppose* $u^{(k)} \rightharpoonup u$ *in* $W^{1,p}(\Omega, \mathbb{R}^n)$, $n < p < \infty$. *Then*

$$\det(Du^{(k)}) \rightharpoonup \det(Du) \quad \textit{in } L^{p/n}(\Omega).$$

Proof. *Step 1.* Main observation is that $\det(Du)$ may be written as a divergence.

$$\begin{aligned}
\det(Du)\delta_{i,j} &= (Du)_{k,i} B_{k,j} \\
\det(Du) &= \frac{1}{n}(Du)_{k,j}(\operatorname{cof} Du)_{k,j} \\
&= \frac{1}{n}\frac{\partial u_k}{\partial x_j}(\operatorname{cof} Du)_{k,j} \\
&= \frac{\partial}{\partial x_j}\left[\frac{1}{n}u_k(\operatorname{cof} Du)_{k,j}\right] \\
&= \operatorname{div}\left[\frac{1}{n}(\operatorname{cof} Du)^T u\right].
\end{aligned}$$

Note that above u_k is the kth component of u, while below and in the statement, $u^{(k)}$ means the kth function of the sequence.

Step 2. It suffices to show that

$$\int_\Omega \eta(x)\det(Du^{(k)})\mathrm{d}x \to \int_\Omega \eta(x)\det(Du)\,\mathrm{d}x$$

for every $\eta \in C_c^\infty(\Omega)$. But by step 1, we have

$$\int_\Omega \eta(x)\det(Du^{(k)})\mathrm{d}x = -\frac{1}{n}\int_\Omega \left(\frac{\partial\eta}{\partial x_l}u_n^{(k)}\right)(\mathrm{cof}(Du^{(k)}))_{m,l}\mathrm{d}x.$$

By Morrey's Inequality, $u^{(k)}$ is uniformly bounded in $C^{0,1-n/p}(\Omega, \mathbb{R}^m)$. By Arzelà-Ascoli's theorem, we may now extract a subsequence $u^{(k_j)}$ that converges uniformly. It must converge to u.

Note that if $f^{(k)} \to f$ uniformly and $g^{(k)} \rightharpoonup g$ in $L^q(\Omega)$,then

$$f^{(k)}g^{(k)} \rightharpoonup f g$$

in $L^q(\Omega)$. Now use induction on dimension of minors.

Alternative: Differential forms calculation:

$$\int_\Omega \eta(x)\det(Du)\,\mathrm{d}x = \int_\Omega \eta(x)\mathrm{d}u_1 \wedge \mathrm{d}u_2 ... \wedge \mathrm{d}u_n = \int_\Omega \eta(x)\mathrm{d}(u_1 \wedge \mathrm{d}u_2 ... \wedge \mathrm{d}u_n)$$

(stopped in mid-deduction, we're supposed do this by ourselves...) □

Theorem 5.14. (Brouwer's Fixed Point Theorem) *Suppose* $u: \bar{B} \to \bar{B}$ *is continuous. Then there is some* $x \in \bar{B}$ *such that* $u(x) = x$.

Theorem 5.15. (No Retract Theorem) *There is no continuous map* $u: \bar{B} \to \partial B$ *such that* $u(x) = x$ *on* ∂B.

Proof. (of Theorem 5.14) Assume $u: \bar{B} \to \bar{B}$ does not have a fixed point. Let $v(x) = u(x) - x$, $v: \bar{B} \to \mathbb{R}^n$. Then $v(x) \neq 0$ and $|v|$ is bounded away from 0. Consider $w(x) = v(x)/|v(x)|$. w is continuous, and

$$w: \bar{B} \to \partial B$$

contradicts the No Retract Theorem. □

Proof. (of Theorem 5.15) *Step 1*. Assume first that u is smooth (C^∞) map from $\bar{B} \to \partial B$, and $u(x) = x$ on ∂B. Let $w(x) = x$ be the identity $\bar{B} \to \bar{B}$. Then $w(x) = x$ on ∂B. But then since the determinant is a null Lagrangian, we have

$$\int_{\bar{B}} \det(Du)\mathrm{d}x = \int_B \det(Dw)\mathrm{d}x = |B|. \tag{5.10}$$

However, $|u(x)|^2 = 1$ for all $x \in B$. That means

$$u_i u_i = 1 \quad \Rightarrow \quad \frac{\partial u_i}{\partial x_j}u_i = 0, \quad j = 1, ..., n.$$

In matrix notation, this is

$$(Du)^T u = 0.$$

Since $|u(x)| = 1$, 0 is an eigenvalue of $Du \Rightarrow \det Du = 0$. This contradicts (5.10).

Step 2. Suppose $u: \bar{B} \to \partial B$ is a *continuous* retract onto ∂B. Extend $u: \mathbb{R}^n \to \mathbb{R}^n$ by setting $u(x) = x$ outside B. Note that $|u(x)| \geqslant 1$ for all x. Let η_ε be a positive , radial mollifier, and consider

$$u_\varepsilon = \eta_\varepsilon * u.$$

$\Rightarrow$ For ε sufficiently small, $|u_\varepsilon(x)| \geqslant 1/2$. Since η_ε is radial, we also have $u_\varepsilon(x) = x$ for $|x| \geqslant 2$. Set

$$w_\varepsilon(x) = \frac{u_\varepsilon(x/2)}{|u_\varepsilon(x/2)|}$$

to obtain a smooth retract onto ∂B contradicting Step 1. □

Remark 5.16. This is closely tied to the notion of the *degree* of a map. Given $u: \bar{B} \to \mathbb{R}$ smooth, we can define

$$\deg(u) = ⨍_B \det(Du) \mathrm{d}x.$$

Note that if $u = x$ on ∂B, then we have

$$\deg(u) = 1 = \deg(\mathrm{Id}).$$

This allows us to define the degree of Sobolev mappings. Suppose $u \in W^{1,1}(\Omega, \mathbb{R}^n)$ with $n < p \leqslant \infty$. Here,

$$\det(Du) = \sum_\sigma (-1)^\sigma \frac{\partial u_1}{\partial x_{\sigma_1}} \cdots \frac{\partial u_n}{\partial x_{\sigma_n}}.$$

So by Hölder's Inequality, $\det(Du) \in L^{p/n} \Rightarrow \det(Du) \in L^1 \Rightarrow$ We can define $\deg(u)$. It turns out that we can always define the degree of *continuous* maps by approximation. Loosely,

1. Mollify $u_\varepsilon = u * \eta_\varepsilon$.
2. Show if u_ε is smooth, then $\deg(u_\varepsilon)$ is an integer
3. $\deg(u_\varepsilon) \to \lim$ as $\varepsilon \to 0$.

$\Rightarrow \deg(u)$ independent of ε for ε small enough.

Reference: Nirenberg, Courant Lecture Notes.

If we know that the degree is defined for continuous maps, then since $p > n$, then $u \in W^{1,p}(B; \mathbb{R}^n)$, $p > n$, we know $u \in C^{0,1-n/p}(B; \mathbb{R}^n)$, so $\deg(u)$ is well-defined.

Question: What happens if $p = n$? Harmonic maps/liquid crystals $u: \Omega \to S^{n-1}$.

Answer: (Brezis, Nirenberg) Don't need u to be continuous to define $\deg(u)$. Sobolev Embedding:

$$W^{1,p} \to \begin{cases} C^{0,1-n/p} & n < p \leqslant \infty, \\ \mathrm{BMO} \supseteq \mathrm{VMO} & p = n, \\ L^q & p < n, q \leqslant p^* = \frac{np}{n-p}. \end{cases}$$

$$[u]_{\mathrm{BMO}} = ⨍_B |u - \bar{u}_B|.$$

VMO: Vanishing mean oscillation.

Theorem 5.17. $\deg \Leftrightarrow \mathrm{VMO}$. *(?)*

(Unfinished business here.)

Weak continuity of determinants: If $u_k \in W^{1,p}(\Omega; \mathbb{R}^n)$ with $n < p$, then if $u_k \rightharpoonup u$, also have

$$\int_\Omega \det(Du_k) \mathrm{d}x \rightharpoonup \int_\Omega \det(Du) \mathrm{d}x$$

$\Rightarrow$ deg is continuous. This is still true if $n = p$, provided we know that $\det(Du_k) \geqslant 0$. (Muller, Bull. AMS 1987)

6 Navier-Stokes Equations

We will briefly write (NSE) for:

$$\begin{aligned} u_t + u \cdot \nabla u &= \underbrace{(\triangle u - \nabla p)}_{\text{force}} + \underbrace{f}_{\text{external force}} \\ \nabla \cdot u &= 0 \\ u(x, 0) &= u_0(x) \text{ given with } \nabla \cdot u_0 = 0 \end{aligned}$$

for $u: [0,\infty)\times\mathbb{R}^n\to\mathbb{R}^n$.

$$(u\cdot\nabla u)_i = u_j\frac{\partial u_i}{\partial x_j};\quad u_t + u\nabla u = \underbrace{\frac{Du}{Dt}}_{\text{material derivative}}.$$

Navier-Stokes v. Euler: RHS has parameter ν

$$u_t + u\cdot\nabla u = -\nu\triangle u - \nabla p.$$

If $\nu = 0$, we have Euler's equations. (Newton's law for fluids) If $\nu\neq 0$, we may as well assume $\nu = 1$.

$\nabla\cdot u = 0$ *is simply conservation of mass:* If the fluid had density ρ, we would have the balance law

$$\partial_t\rho + \operatorname{div}(\rho u) = \partial_t\rho + (\nabla\cdot u)\rho + u\cdot\nabla\rho = 0.$$

If we further assume

$$\partial_t\rho + u\cdot\nabla\rho = 0,$$

that is

$$\frac{D\rho}{Dt} = 0,$$

then we have $\nabla\cdot u = 0$. Compare with Burgers Equation:

$$\partial_t u + u\,\partial_x u = 0,\quad x\in\mathbb{R}, t>0.$$

It is clear that singularities form for most smooth initial data.

The pressure has the role of maintaining incompressibility. Take the divergence of (NSE1):

$$\nabla\cdot(\cancel{\partial_t}u + u\cdot\nabla u) = \nabla\cdot(-\nabla p + \cancel{\triangle}u).$$

Then

$$\operatorname{Tr}(\nabla u^T\nabla u) = -\triangle p.$$

Thus $-\triangle p\geqslant 0$. Flows are steady if they don't depend on t. In this case we have

$$\begin{aligned} u\cdot\nabla u + \nabla p &= \triangle u,\\ \nabla\cdot u &= 0. \end{aligned}$$

If $\nu = 0$, we have ideal (i.e. no viscosity), steady flows:

$$u\cdot\nabla u + \nabla p = 0,\quad \nabla\cdot u = 0 \Rightarrow \nabla\left(\frac{u^2}{2} + p\right) = 0,\quad \nabla\cdot u = 0,$$

or $|u|^2/2 + p = \text{const}$, which is called *Bernoulli's Theorem*.

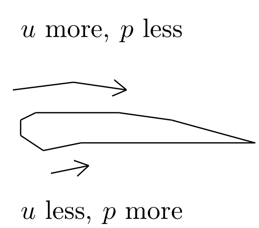

Figure 6.1.

Vorticity: $\omega = \operatorname{curl} u$. This is a scalar when $n = 2$.

Vorticity equation:

$$\begin{aligned} \partial_t\omega + \nabla\times(u\cdot\nabla u) &= \triangle\omega,\\ \nabla\cdot u &= 0,\\ \nabla\times u &= \omega. \end{aligned}$$

In 2-D, this is simply

$$\begin{cases} \partial_t \omega + u \cdot \nabla u = \triangle \omega, \\ \begin{cases} \nabla u = 0, \\ \nabla \times u = \omega, \end{cases} \end{cases}$$

where the first equation is an advection-diffusion equation for ω.

6.1 Energy Inequality

Assume $f \equiv 0$ for simplicity. Dot the first NSE above with u:

$$\frac{\partial}{\partial t}\left(\frac{|u|^2}{2}\right) + u \cdot \nabla\left(\frac{|u|^2}{2} + p\right) = \nabla \cdot (u \cdot \nabla u) - |\nabla u|^2.$$

Integrate over $\mathbb{R}^n$:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^n} \frac{|u|^2}{2} \mathrm{d}x = -\int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x \quad \Rightarrow \quad \|u(\,\cdot\,, t)\|_{L^2}^2 \leqslant \|u_0\|_{L^2}^2.$$

$$\int_0^t \int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x \leqslant \|u_0\|_{L^2}^2.$$

Theorem 6.1. (Leray, Hopf) *For every $u_0 \in L^2(\mathbb{R}^n)$, there exist distributional solutions $u \in L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^n))$, such that the energy inequalities hold.*

Q: Regularity/Uniqueness of these solutions? $n = 2$, Ladyzhenskaya $\rightarrow$ uniqueness.

6.2 Existence through Hopf

Reference: Hopf's paper on website, Serrin's commentary.

$$\begin{aligned} \partial_t u + u \cdot \nabla u &= -\nabla p + \triangle u, \\ \nabla \cdot u &= 0. \end{aligned}$$

$x \in G$ = open subset of $\mathbb{R}^n$, $\hat{G} = G \times (0, \infty)$ space-time. Initial boundary value problem:

$$u(x, 0) = u_0(x) \quad \text{given and} \quad \nabla \cdot u_0 = 0.$$

No-slip boundary conditions:

$$u(x, t) = 0 \quad \text{for} \quad x \in \partial G.$$

(Compare this to Euler's equation, where we only assume that there is no normal velocity.)

6.2.1 Helmholtz projection

Recall the example of a divergence-free vector field from the last final.

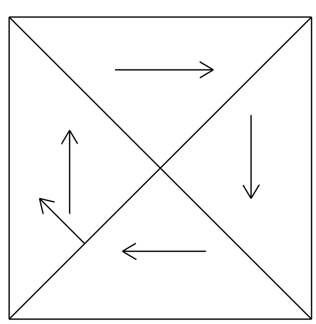

Figure 6.2.

Observe that only the continuous boundary-normal field matters, not the (discontinuous) boundary-tangential field. We want to push the requirement $\nabla \cdot u = 0$ into L^2.

$\nabla \cdot u = 0$ in $\mathcal{D}'$ simply means

$$\int_G u \cdot \nabla \varphi \, \mathrm{d}x = 0$$

for every $\varphi \in C_c^\infty(\Omega)$. Let $P = \text{closure}\,\{\nabla\varphi \colon \varphi \in C_c^\infty \text{ in } L^2(G, \mathbb{R}^n)\}$. P is the space of gradients in $L^2(G)$. If $h \in P$, then there exists $\varphi_k \in C_c^\infty(G)$ such that $\nabla\varphi_k \to h$ in $L^2(G, \mathbb{R}^n)$. Then

$$L^2(G) = \underbrace{P}_{\text{gradients}} \oplus \underbrace{P^\perp}_{\text{divergence-free}}.$$

6.2.2 Weak Formulation

In all that follows, $a \in C_c^\infty(\hat{G}, \mathbb{R}^n)$ is a divergence-free vector field

$$\partial_t u + \underbrace{u \cdot \nabla u}_{\text{read as } \frac{1}{2}\nabla\cdot(u\otimes u)} = -\nabla p + \triangle u.$$

In coordinates,

$$\partial_t u_i + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial^2 u_i}{\partial x_j \partial x_j} \quad i = 1, \ldots, n.$$

Take inner product with a and integrate by parts:

$$(W_1) \quad -\int_{\hat{G}} \left[\partial_t a \cdot u + \underbrace{\nabla a \cdot (u \otimes u)}_{\text{here we use:}} + \triangle a \cdot u\right] \mathrm{d}x\,\mathrm{d}t = 0$$

$$\begin{aligned}\int_{\hat{G}} a_i u_j \frac{\partial u_i}{\partial x_j} \mathrm{d}x\,\mathrm{d}t &= -\int_{\hat{G}} \frac{\partial}{\partial x_j}(a_i u_j) u_i \,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_{\hat{G}} \frac{\partial a_i}{\partial x_j} u_j u_i \,\mathrm{d}x\,\mathrm{d}t - \int_{\hat{G}} a_i \cancel{\frac{\partial u_j}{\partial x_j}} u_i \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

For the weak form, consider that

$$\int_{\hat{G}} a \cdot \nabla p = -\int_{\hat{G}} (\operatorname{div} a) p \,\mathrm{d}x\,\mathrm{d}t = 0$$

means we lose the pressure term. Also, recall

$$u \otimes u := u_i u_j = u\,u^T.$$

If $A, B \in \mathbb{M}^{n\times n}$, then $A \cdot B = \operatorname{tr}[A^T B]$. Similarly, weak form of $\nabla u = 0$ is

$$(W_2) \quad \int_{\hat{G}} u \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t = 0 \quad \text{for every } \varphi \in C_c^\infty(\hat{G}).$$

Definition 6.2. $V = \text{closure}\,\{a \in C_c^\infty(\hat{G}, \mathbb{R}^n), \nabla \cdot a = 0\}$ *w.r.t. the space time norm*

$$\begin{aligned}\|a\|_V &= \int_0^\infty \int_G (|a|^2 + |\nabla a|^2)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\hat{G}} \left[a_i a_i + \frac{\partial a_i}{\partial x_j}\frac{\partial a_i}{\partial x_j}\right] \mathrm{d}x\,\mathrm{d}t\end{aligned}$$

Space for initial conditions:

$$L_0^2(G, \mathbb{R}^n) = \text{closure}\{b \in C_c^\infty(G, \mathbb{R}^n)\}$$

in $L^2(G, \mathbb{R}^n)$. Observe that by the Helmholtz projection,

$$L_0^2(G, \mathbb{R}^n) = \underbrace{P_0}_{\text{gradients}} \oplus \underbrace{P_0^\perp}_{\text{divergence free vector fields with zero BC}}.$$

Theorem 6.3. (Leray, Hopf) *Let $G \subset \mathbb{R}^n$ be open. Suppose $u_0 \in P_0^{\perp}(G)$. Then there exists a vector field $u \in V$ that satisfies the weak form (W_1), (W_2) of the Navier-Stokes equations. Moreover,*

- $\|u(t, \cdot) - u_0\|_{L^2(G)} \to 0$ *as* $t \downarrow 0$.
- *Energy inequality*
$$\frac{1}{2} \int_G |u(x,t)|^2 \mathrm{d}x + \int_0^t \int_G |\nabla u(x,s)|^2 \,\mathrm{d}x \,\mathrm{d}s \leqslant \frac{1}{2} \int |u_0(x)|^2 \mathrm{d}x$$
for $t > 0$.

Remark 6.4.

1. No assumptions on smoothness of ∂G.
2. No assumptions on space dimension.

(Yet there is a large gap between $n = 2$ and $n > 2$.)