PDE Summary

1 General Stuff

- **Standard mollifier:**
  \[ \eta(x) = \exp\left(\frac{1}{x^2 - 1}\right)1_{[-1,1]} \]
  is a \(C^\infty\) hump.

\[ \eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \eta(x/\varepsilon). \]

Normalization (\( \int = 1 \)) is still missing.

- **Gamma function:**
  \[ \Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt. \]

- **Volumes of sphere and ball:**
  \[ |S^{n-1}| = \omega_n r^{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}r^{n-1}. \]
  \[ |B^n| = \omega_n r^n. \]

- **Green’s identities:**
  \[ \int_U v\Delta u = -\int_U \nabla v \cdot \nabla u + \int_{\partial U} v\partial_n u \]
  \[ \int_U v\Delta u - u\Delta v = \int_{\partial U} v\partial_n u - u\partial_n v \]

- **Young’s Inequality:**
  \[ \|f \ast g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q} \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}. \]
  In particular \( q = 1, r = p \).

- **Generalized Hölder:**
  \[ \|f_1 \cdot f_2 \cdots f_m\|_{L^1} \leq \|f_1\|_{p_1}\|f_2\|_{p_2} \cdots \|f_m\|_{p_m} \]
  if
  \[ \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = 1. \]

- **Interpolation Inequality for \(L^p\):** If \( 1 \leq s \leq r \leq t \leq \infty \)
  \[ \frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{t}, \]
  \( u \in L^s \cap L^t \), then \( u \in L^r \) and
  \[ \|u\|_{L^r} \leq \|u\|^\theta_{L^s} + \|u\|^{1-\theta}_{L^t}. \]

- **Compact:** Every open cover has finite subcover. Metric space: \( \Leftrightarrow \) sequentially compact. Heine-Borel (finite-dim): \( \Leftrightarrow \) closed and bounded.

- **Arzelà-Ascoli:** \((S, d)\) compact metric space. \( M \subset C(S) \) with sup-norm is compact if \( M \) is bounded, closed and equicontinuous.

- **Precompact:** has compact closure.

- **Compact operator:** \( T: B_1 \rightarrow B_2 \) compact if \( T \) continuous and \( T(A) \) precompact for every bounded \( A \).

- **Fredholm Alternative:** \( T: B \rightarrow B \) linear, continuous, compact:
  - either \((I - T)x = 0\) has a nontrivial solution
or \((I - T)^{-1}\) exists and is bounded.

“Uniqueness and Compactness ⇒ Existence”.

- Lax-Milgram: \(B: H \times H \to \mathbb{F}\), bounded above and coercive ⇒ \(B[u, g] = F(u)\) solvable in \(H\) for every \(g \in H\).
  
  Proof: Build operator \(T_g: H \to H^*\) that gives \(T_g(u) = B[u, g]\) (Riesz rep.). Prove 1-1 and onto.
  
  The point is: no symmetry.

- Banach-Steinhaus/Uniform Boundedness Principle:
  
  \(X BR, Y NR, T_i \in L(X,Y) (i \in I), \sup_{i \in I} \|T_i x\| < \infty (x \in X) \Rightarrow \sup_i \|T_i\| < \infty\).
  
  Read as “linear+pw bounded ⇒ uniformly bounded.”

2 Equations

- Classification of second order equations:
  
  \[ A_{ij} \partial_i \partial_j u + B_i \partial_i u + C = 0, \]

  where \(A\) is symmetric WLOG can be rewritten into one of

  \[ u_{xx} + u_{yy} + \text{l.o.d.} = F, \]
  \[ u_{xx} - u_{yy} + \text{l.o.d.} = F, \]
  \[ u_{xx} \pm u_y + \text{l.o.d.} = F. \]

- Minimal surface equation:
  
  \[ \text{div} \left( \frac{D u}{\sqrt{|D u|^2 + 1}} \right) = 0 \]

- Monge-Ampère equation:
  
  \[ \det(D^2 u) = K(x)(1 + |D u|^2)^{(n+2)/2} \]

3 Laplace’s Equation

\(U\) open.

- \(u \in C^2(U)\): harmonic, subharmonic \(\Delta u \geq 0\), superharmonic.

- Mean Value Inequality: \(u\) subharmonic
  
  \[ u(x) \leq \int_{S(x,r)} u(y) dS_y \]
  \[ u(x) \leq \int_{S(x,r)} u(y) dB \]

  (implies Mean Value Property if harmonic)

  Proof: \(0 \leq \int_B \Delta u = \int_S \partial_u u, \) then exploit \(\partial u \partial(x + pm) . \int_B u = \int_r \int_{|u|=1} u = u \int_r \).

- Strong Maximum Principle: \(U\) bounded, connected, \(u\) subharmonic, \(u(x) = \sup_{U} u \Rightarrow u\) constant

  Proof: Consider \(\{u = \sup\}\). By MVI, \(u = \sup\) on any ball in \(U\). Thus \(\{u = \sup\}\) open. But so is \(\{u < \sup\}\). \(U = \{u = \sup\} \cup \{u < \sup\}\), both open ⇒ \(\{u = \sup\} = U\).

- Weak maximum principle: \(u \in C(\bar{U})\) and subharmonic. Then \(u\) assumes extrema on the boundary.

  Proof: SMP or: Suppose \(x \in U\) is max and \(\Delta u > 0\). Then \(Du = 0\) and \(D^2 u\) negative semidef, contradicting \(\Delta u = \text{tr}(D^2 u) \geq 0\). If only \(\Delta u \geq 0\), consider \(u + \varepsilon |x|^2\), which is strictly subharmonic.

- Strong ⇒ constant, Weak ⇒ extrema on boundary.
Laplace’s Equation

- **Uniqueness** follows directly from the WMP.
- **Harnack’s Inequality**: $u \geq 0$ (!) harmonic, $U' \subset \subset U$ connected $\Rightarrow \exists C$ such that $\sup u < C \inf u$.
  Proof: Pick $x_1, x_2 \in U$, apply MVP for large and small circle, respectively, then shrink/expand domain by using $u \geq 0$, take sup/inf. Use cover of balls to repeat argument as necessary.
- **Fundamental solution**: look for radial symmetry
  \[
  \psi = C + \begin{cases} 
  \frac{1}{2\pi} \log r & n = 2, \\
  \frac{1}{(2 - n)\omega_n} r^{2-n} & n \geq 3.
  \end{cases}
  \]
  Constant chosen because it gives the right constant to prove $\Delta \psi = \delta_0$ (use Green’s second id on a ball surrounding the singularity). $K(x, \xi) = \psi(|x - \xi|)$.
- **Liouville’s Theorem**: (only in 2D) Subharmonic functions bounded above are constant.
- For $u \in C^2(U)$:
  \[
  u(\xi) = \int_U K(x, \xi) \Delta u dx + \int_{\partial U} u \partial_n K(x, \xi) - K(x, \xi) \partial_n u dS_x. 
  \]  \(1\)
  Proof: Integrate on $U \setminus B_\varepsilon, \varepsilon \to 0$.
  Remains valid if $K$ replaced by $K + w$ with harmonic $w$.
- **Green’s function for Dirichlet problem**: $\Delta \xi G = \delta_\xi, G(x, \xi) = 0$ for $x \in \partial U$. Use $G$ in (1). To get one, we need to find $w$ with $w = -K$ on $\partial U$. (Use method of images.) For a ball, we get the Poisson kernel
  \[
  H(x, \xi) = \frac{r^2 - |\xi|^2}{\omega_n r |x - \xi|^n}
  \]
  Poisson’s integral formula:
  \[
  u(\xi) = \int_{S(0, r)} H(x, \xi) f(x) dS_\xi.
  \]
- **Kelvin’s transformation**: $u$ harmonic $\Rightarrow$ $|x|^{2-n} u(x/|x|^2)$ harmonic for $x \neq 0$.
- Properties of $H$:
  - $H(x, \xi) = H(\xi, x)$
  - $H(x, \xi) > 0$ on $B(0, r)$
  - $\Delta \xi H(x, \xi) = 0$ for $\xi \in B(0, r)$ and $x \in S(0, r)$
  - $\int_{S(0, 1)} H(x, \xi) dS_x = 1$
- **Existence on a ball**: also gives $C(\bar{B})$
  Proof: Differentiate under integral (using DCT). Prove continuity onto the boundary by
  \[
  u(\xi) - f(y) = \int_{S(\xi, r)} H(x, \xi) (f(x) - f(y)) dS_x
  \]
  Use $\varepsilon$-$\delta$-continuity of $f$ and split integral into $|x - y| < \delta$ and $|x - y| > \delta$. (Method called approximate identities.)
- **Converse of MVP**: $u \in C(U)$ harmonic $\Leftrightarrow$ satisfies MVP for every $B(x, r) \subset U$.
  Proof: Construct a harmonic function $v$ on $B(x, r)$ with $v = u$ on $S(x, r)$. $v - u$ satisfies MVP on any subcircle, thus it satisfies the strong maximum principle. Thus $v = u$.
- **Real analytic**: completely represented by absolutely convergent Taylor series.
  \[
  \exists M > 0: |\partial^n f(y)| \leq \frac{M |\alpha|^n}{r^{n+1}} \Leftrightarrow \text{analytic}.
  \]
  Real analytic $f$ is completely determined by power series (use similar open-set method on $\{\partial^n h(y) = 0 \forall \alpha\}$ as SMP)
• **Harmonic ⇒ Analytic:** Consider $H(x, \xi + i\sigma)$. Find a region of $\sigma$ where $H$ is differentiable.

• Analyticity estimates can be obtained by the MVP applied to $\partial_x u$, then coordinatewise Gauß, giving

$$|\partial_x u(x)| \leq \frac{n}{r} \max_{S(x,r)} |u| \leq \frac{n}{r} \sup_U |u|.$$  

Then iterate this estimate with $1/|\alpha|$ radius to get

$$|\partial^\alpha u(x)| \leq \left( \frac{n|\alpha|}{r} \right)^{|\alpha|} \max_{S(x,r)} |u|.$$  

• Uniformly (on compact subsets of $U$) converging sequences of harmonic functions converge to harmonic functions.

Proof: Limit is continuous (because of uniform convergence). Now exchange limits (DCT) in MVP and prove harmonicity.

• **Harnack’s convergence theorem:** $u_k$ harmonic, increasing and bounded at a point. Then $(u_k)$ converges uniformly on compact subsets to a harmonic function.

Proof: above + Harnack inequality.

• “Montel’s Theorem”—a compactness criterion:

$(u_k)$ bounded, harmonic ⇒ $\exists$ uniformly (on compact subsets) converging subsequence $\to$ harmonic limit.

Proof: $(u_k)$ is equicontinuous because of the derivative estimates and the assumed uniform bound.

• **Subharmonicity on $C(U)$:** Satisfies MVI locally.

• Perron’s method:

  o $S_f := \{ v \in C(\overline{U}), v \leq BC, v$ subharmonic $\}$.  
  o $u := \sup S_f$ is harmonic.

  Proof:

  - $S_f$ is closed under finite max. (MVI)
  - Harmonic lifting: $v$ subharmonic,

    $$V(x) = \begin{cases} \text{harmonic function with matching BCs } B(\xi, r), & v \in S_f \\ \text{elsewhere.} & \end{cases}$$

    $v \in S_f \Rightarrow V \in S_f, v \leq V$.

    - Fix a closed ball, grab sequence $v_k \to u$ at a point $\xi$. $\bar{v}_k := \max(v_1, \ldots, v_k, \min BC)$.

    - Replace these by their harmonic lifting $V_k$ around $\xi$.

    - HCT for a limit $V$.

    - Prove $V = u$ on ball by finding SMP uniqueness of harmonic liftings of in-between ($V < u$) functions.

• **Barrier function at $y \in \partial U$/regular boundary point:**

  $w \in C(\overline{U})$ subharmonic, $w(y) = 0$, $w(\partial U \setminus \{y\}) < 0$.

  $\exists$ tangent plane $\Rightarrow$ regular

  $\exists$ exterior sphere $\Rightarrow$ barrier $= K$(boundary point, outside center) $- K(x, outside center)$

  $\exists$ exterior cone $\Rightarrow$ regular

• At regular boundary points, $u = BC$.

Proof:

  o Fix $\varepsilon > 0$. $\delta$ from $\varepsilon$-$\delta$ with $f$.

  o $v = BC + A \cdot \text{barrier} - \varepsilon$, where $A w \leq -2 \max BC$ outside a ball around the boundary point in question. $v$ subharmonic by def.
Show $v \leq f(x)$ on boundary and interior.

Do some funky tricks involving $-f$, its Perron function, and the maximum principle to show opposite inequality.

- The Dirichlet problem is solvable for all continuous BC data iff the domain is regular.

### 3.1 Energy Methods

- $0 = \int w \Delta w = \int |\nabla w|^2$ proves uniqueness in $C^2(\bar{U})$.

- **Energy Functional:**
  \[ I[w] = \int_{\bar{U}} \frac{1}{2} |\nabla w|^2 + wg dx \]

  for $g$ the RHS.

- **Dirichlet’s principle:** $u \in C^2(\bar{U})$ solves PDE+BC $\iff$ it minimizes $I[u]$ over \{ $w \in C^2(\bar{U}) \land w = \text{RHS on } \partial \Omega$ \}.

  Proof: PDE $\Rightarrow$ min: Start from
  \[ 0 = \int (-\Delta u + g)(u - w), \]

  use Gauß, Cauchy-Schwarz, $\sqrt{a} \sqrt{b} \leq 1/2 (a^2 + b^2)$.

  $\min \Rightarrow$ PDE: $w = u + tv$, for $v \in C^\infty_c$. Differentiate by $t$.

### 3.2 Potentials

- **Potential of a measure:**
  \[ u_{\mu}(x) = \frac{2-n}{\omega_n} \int_{\mathbb{R}^n} K(x, y) \mu(dy) = \int_{\mathbb{R}^n} |x - y|^{2-n} \mu(dy) \]

- Computable for a sphere with uniform charge density (same as point charge), finite line, disk.

- $u_{\mu} = 0 \Rightarrow \mu = 0$.

  Proof: Show $\mu * f = 0$ for any $f \in C^\infty_c$ by
  \[ \mu * f = \mu * (K * \Delta f) = (\mu * K) * \Delta f = 0. \]

- **Potentials of compact set:** Harmonic function with BC 1 on compact set $F$ and BC zero at infinity. Perron function on ever-increasing balls–independent of exact domains.

- **A (unique) generating (positive) measure on $\partial F$ exists:**

  Proof (if $\partial F \in C^2$): by Poisson’s boundary representation formula (with both $u$ and $\partial_n u$)
  \[ p_F(\xi) = \int_{\partial F} K(x, \xi) \partial_n p_F dS_x, \]

  $\partial_n u \leq 0$ by the max principle (1 on the boundary must be the max value) $\Rightarrow$ positivity.

  Proof (if not):
  - Approximate $F$ through shrinking compact sets with $C^\infty$ boundary ($1/k^2$-mollified indicators of $F^{1/k} = \{ \text{dist}(x, F) \leq 1/k \}$). $\psi = \varphi_{1/k} * 1_{F^{1/k}}$. Then consider $F^{1/2k} \subset \psi^{-1}([c, 1]) \subset F^{1/k}$ and use Sard’s Theorem to deduce boundary smoothness for a.e. $c$. Generate $\mu_k$ by above theorem.
  - $p_{F_k} \rightarrow p_F$ uniformly on compact subsets (Harnack)
  - Prove $\mu_k(\mathbb{R}^n) \leq R^{n-2}$ by using a $B(0, R) \supset F_k$–use Fubini and the generator of the disk potential. (“Gauß’ trick”) Thus $\exists$ weak-* convergent subsequence supported on $\partial F$. Thus convergenc of $p_{F_k} \rightarrow p_F$ away from $\partial F$. Uniqueness by uniqueness of potentials of measures.
3.3 Lebesgue’s Thorn
• In 2D, Riemann mapping theorem guarantees that point regularity is topological, not geometric.
• Lebesgue’s Thorn: Using level sets of the potential of the measure $x^\beta dx$ on $(0, 1)$, one may construct exceptional points.

3.4 Capacity
• $\text{cap}(F) = \mu_F(\mathbb{R}^n) = \frac{2-n}{\omega_n} \int_{\partial F} \mu_{\partial F} dS_x$.
• If $\partial F \in C^2$, Green’s 1st id gives $\text{cap}(F) = \frac{2-n}{\omega_n} \int_{U \subset \mathbb{R}^n \setminus F} |\nabla p_F|^2$.
• Wiener’s criterion: $y \in \partial U$ regular $\iff$
  \[ \lambda^{2-n} \sum_{k=0}^{\infty} \lambda^{k(2-n)} \text{cap}(F_k) \quad F_k := \{ \lambda^{k+1} \leq |x-y| \leq \lambda^k \} \quad (\lambda \in (0, 1)). \]
• Properties of capacity:
  ○ $F_1 \subset F_2 \Rightarrow \text{cap}(F_1) \leq \text{cap}(F_2)$ (Gauß’ Trick!)
  \[ \text{cap}(F_1) = \int_{\mathbb{R}^n} \mu_1(dx) = \int_{\mathbb{R}^n} p_2 \mu_1(dx) = \int \int |x-y|^{2-n} \mu_2(dy) \mu_1(dy) = \int p_1 \mu_2(dy) \leq \text{cap}(F_2). \]
  ○ $(F_k)$ nested sequence with $\bigcap F_k = F$, then $\text{cap}(F_k) \to \text{cap}(F)$.
    (smooth $\varphi = 1$ on $F_1$, $\text{cap}(F) = \int \varphi \mu_F \leftarrow \int \varphi \mu_{F_k} = \text{cap}(F_k)$)
  ○ $\text{cap}(A \cup B) \leq \text{cap}(A) + \text{cap}(B)$.
    ($p_A \leq p_A + p_B$ by WMP. Then use Gauß’ trick.)
  ○ $\text{cap}(A \cup B) + \text{cap}(A \cap B) \leq \text{cap}(A) + \text{cap}(B)$
• $\text{cap}(B(0, R)) = \text{cap}(S(0, R)) = R^{n-2}$.
• Screening: nested spheres $A \subset B$. $\text{cap}(A \cup B) = \text{cap}(B)$ (think of the potentials)
• $\text{cap}(F) = \sup \{ \mu(F) : \text{supp}(\mu) \subset F, u_\mu(F) \leq 1 \}$ (Smooth approx $F_k$ to $F$ so that $p_{F_k} = 1$ on $\partial F$. Then Gauß’ trick.)
• Coulomb energy:
  \[ E[\mu] = \frac{1}{2} \int \int |x-y|^{2-n} \mu(dx) \mu(dy). \]
  Mutual energy:
  \[ E[\mu, \nu] = \frac{1}{2} \int \int |x-y|^{2-n} \mu(dx) \nu(dy). \]
• Properties:
  ○ If $E[|\mu|] < \infty$, then pos.def.
  ○ CSU
  ○ $\mu \mapsto E[\mu]$ strictly convex
• Gauß’ principle: $\mu \geq 0$ finite measure on $F$.
  \[ G[\mu] = E[\mu] - \mu(F) \geq -\frac{1}{2} \text{cap}(F) \]
Proof:

- $G(\mu)$ bounded below ($F$ compact $\Rightarrow$ $|x - y|$ bdd.)
- Infinimizing sequences are precompact (i.e. have bounded $\mu_k(F)$)
- $G$ is wloc. (take infimizing sequence $(\mu_k)$, use max $(M, |x - y|)$ to cut off, $k \to \infty$, $M \to \infty$ (MCT), consider $E[\mu - \mu_k]$)
- Minimizer is unique (strict convexity)
- Minimizer is $\mu_F$ (Consider Euler-Lagrange Equation)
- Evaluate minimum

- **Kelvin’s principle:**
  \[
  \frac{1}{2\text{cap}(F)} = \inf \{ E[\mu]; \mu \geq 0, \text{supp}(\mu) \subset F, \mu(F) = 1 \}.
  \]

Proof: Apply Gauß’ principle to $t\mu$, choose $t = \text{cap}(F)$.

4 Heat Equation

- **Conservation of mass:** $\partial_t u + \text{div}(v) = 0$
- **Fick’s law:** $v = -\alpha^2 \nabla u$.
- Together: $u_t = \Delta u$
- Parabolic scaling invariance: $x \mapsto \lambda x$, $t \mapsto \lambda^2 t$.
- Use conservation of mass ($\partial_t \int u = 0$) to obtain the ansatz $u(x, t) = t^{-n/2} g\left(r t^{-1/2}\right)$. Plug in heat equation to get the heat kernel
  \[
  k(x, t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}.
  \]
- Use
  \[
  2 \int_{y > a} e^{-y^2} dy < 2 \int_{y > a} \frac{y}{a} e^{-y^2} = \frac{e^{-a^2}}{a}.
  \]
  and in-boxing the ball to show
  \[
  \int_{|x| \geq \delta} k(x, t) dx \to 0 \quad \text{as} \quad t \to 0.
  \]
- $u = k * f$ solves $u_t = \Delta u$ for $u \to f$ for $t \to 0$.
- **Tychonoff counterexample** for uniqueness:
  \[
  u(x, t) = \sum_k g_k(t) x^{2k}
  \]
- **Widder’s Theorem:** $u \geq 0$ $\Rightarrow$ uniqueness.
- **Heat ball:** $E(x, t, r) = \{k(x - y, t - r) \geq r^{-n}\}$.
- $V_T = U \times [0, T]$, $\partial_1 V_T = \text{all except top “lid”}$, $\partial_2 V_T = \text{lid}$.
- **Mean Value Property:** $u \in C^2(V_T)$, $\partial_t u - \Delta u \leq 0$, $E(\ldots) \subset V_T$:
  \[
  u(x, t) \leq \frac{1}{4\pi^n} \int_{E(x, t, r)} \int u(y, s) \frac{|x - y|^2}{(t - s)^2} dy ds
  \]
  - Exists for heat spheres as well.
Converse: Equality and $C^2(V_T)$ implies $\partial_t u = \Delta u$.

Proof: Let RHS $= \varphi(r)$.

$\varphi(0) = u(x, t)$,

$$\varphi'(r) = -C \int (\partial_s u - \Delta u) \psi \, ds \geq 0$$

with $\{ \psi \geq 0 \} = E(\ldots)$.

- **Strong Maximum Principle:** $U$ open, bounded, connected, $u \in C(\bar{V}_T)$ and satisfies MVI. Then
  $$\max_{\bar{V}_T} u \leq \max_{\partial_1 V_T} u.$$

  If max attained at $(x, t) \in V_T$, then $u$ is constant in $\bar{V}_T$.

  Proof: If max attained in interior, then $u = M$ on heat ball. Then a polygonal path reaches every point on $V_T$.

- Temperatures are analytic:
  - Green’s functions for the heat equation:
  - **Strong Converse of MVP.**
  
  Proof: Construct parallel solution by Green’s functions. Conclude uniqueness by MVP.

4.1 Difference Schemes and Probabilistic Interpretation

- Work on a lattice.
  - **Strong Maximum Principle** (subharmonic $\Rightarrow$ assume max $M$ in interior $\Rightarrow M = u \leq E[x + h\omega] \leq M$.)
  - Implies discrete Laplacian has trivial null-space $\Rightarrow \exists !$
  - Allows Discrete Poisson Integral Formula. (by solving for $\delta$ on the boundary)
  - **Markov property:** $E[X_{m+1}|X_1, \ldots, X_m] = E[X_{m+1}|X_m]$.
  - (Super)Martingale property: $u$ subharmonic $\Rightarrow E[u(X_{m+1})|X_m] \geq u(X_m)$ (just like discrete SMP) [with $X_m$ a random walk]
  - **Strong Martingale Property:** $m$ may be a stopping time.
  - If $M_U$ is first passage time to $\partial U$, then $u = E[f(x + W_{M_U})]$ (f=BC, $u$ harmonic)

  $$E[f(x + W_{M_U})] = \sum_{y \in \partial U_h} H(x, y) f(y) = \sum_{y \in \partial U_h} \int P(\text{hit } y) f(y).$$

- Method of relaxation:
  $$u^{(l+1)}(x) = \text{avg}(u^{(l)}) \text{ on pixels surrounding } x$$

- Brownian motion: Same formula as above holds for continuous-time. (Central Limit Theorem, path space version of it, $W_t \sim k(x, t/2)$. Cylinder sets. Convergence in weak-* topology. Law of iterated logarithm. Proof of CLT: Convolution of densities becomes multiplication after Fourier transform. Use independence. Done.)

- Feynman-Kac formula: $u_t = \frac{1}{2} \Delta u$ with IC $f$.

  $$E(f(x + W_t)) = u(x, t)$$

- Implications on boundary regularity:
  - $u$ defined by F-K is the Perron function
  - $y \in \partial U$ is regular iff $P(T_y = 0) = 1$ (BM immediately exits $U$.)
  - Littlewood’s crocodile
  - Lebesgue’s thorn
4.2 Hearing the shape of a drum

- **Spectral measure:**
  \[ A(\lambda) = \sum_{k=1}^{\infty} 1_{\lambda_k \leq \lambda}(\lambda). \]

- **Weyl’s result:**
  \[ \lim_{\lambda \to \infty} A(\lambda) = \frac{|U|}{(2\pi)^{n/2} \Gamma(n/2)}. \]

- **Kac’s result:**
  \[ \lim_{t \to 0^+} (2\pi t)^{n/2} \sum_{k=1}^{\infty} e^{-\lambda_k t} = (2\pi t)^{n/2} \int e^{-t\lambda} A(d\lambda) = |U|. \]

(Weyl ⇒ Kac: Integrate by parts, rescale. Proof of Kac: represent Green’s function in terms of eigenfunctions somehow.)

5 Wave equation

- \[ u_{tt} = c^2 u_{xx} \]

- **D’Alembert’s formula:**
  \[ u(x, t) = \frac{1}{2} \left[ f(x + ct) + f(x - ct) + \frac{1}{c} \int_{x-ct}^{x+ct} g(y) dy \right]. \]

- **Characteristics.**
- **Parallelogram identity:**
  \[ u(\text{top}) + u(\text{bottom}) = u(\text{left}) + u(\text{right}). \]


- **D’Alembertian:**
  \[ \Box u := u_{tt} - c^2 \Delta u = 0. \quad u = f, \quad u_t = g. \]

- **Fourier Analysis:**
  \[ \hat{u}(\xi, t) = \hat{f}(\xi) \cos(c|\xi|t) + \hat{g}(\xi) \sin(c|\xi|t)/|\xi|t = \hat{f}(\xi) \cos(c|\xi|t) + \hat{g}(\xi) \partial_t \cos(c|\xi|t): \]
  \[ u(x, t) = \int_{\mathbb{R}^n} k(x - y, t) g(y) dy + \partial_t \int_{\mathbb{R}^n} k(x - y, t) f(y) dy \]

  Needs to coincide with solution formula.

- For \( n = 3, \) \( k = t \cdot \) uniform measure on \( \{|x| = ct\} \)

- **Method of Spherical means:**
  Observe:
  \[ M_u(x, r) = \int_{S(x, r)} u(y) dS_y \]

  satisfies Darboux’s Equation:
  \[ \Delta_x M_u = "\Delta_r" M_u = \left( \partial_{rr} - \frac{n-1}{r} \partial_r \right) M_u. \]

  Similarly, if \( u \) solves \( u_{tt} = u_{xx} \), then \( M_u \) solves the Euler-Poisson-Darboux equation:
  \[ (M_u)_{tt} - \Delta_r M_u = 0. \]

  In 3D, this reduces the wave equation to \( \partial_t^2 (r M_u) = \partial_r^2 (r M_u) \), which we can solve by D’Alembert’s formula for all \( x \). Then
  \[ u = \lim_{r \to 0} M_u(r), \]
  \[ \frac{1}{(2\pi)^{n/2}} \int_{|y|=ct} e^{-i\xi \cdot y} dS_y = \frac{\sin(c|\xi|t)}{|c|\xi}. \]
• **Huygens’ principle.**

• **Hadamard’s method of descent:** Treat 2D equation as 3D equation, independent of third coordinate.

• **General solution for odd** \(n \geq 3\): Assume \(u'(0) = 0\). Define

\[
v(x, t) := \int k(s, t)u(x, s)ds
\]

as a temporal heat kernel average. Oddly, \(\partial_t v = \Delta_x v\). Solve this. Rewrite using spherical means. Change variables as \(\lambda = 1/4t\) and invert using the Laplace transform

\[
h^\#(\lambda) = \int_0^\infty e^{-\lambda \xi} h(\varphi) d\varphi.
\]

• Uniqueness by energy norm.

### 6 Distributions/Fourier Transform

\(U \subset \mathbb{R}^n\) open

- **\(\mathcal{D}(U) := C_0^\infty(U)\).** \(\varphi_k \to \varphi\) iff
  - \(\exists\) fixed compact set \(F\): \(\text{supp}(\varphi_k) \subset F\)
  - \(\forall \alpha: \sup_F |\partial^\alpha \varphi_k - \partial^\alpha \varphi| \to 0\).

- **Distribution:** \(\mathcal{D}'(U)\)
  - Convergence: \(L^k \overset{\mathcal{D}}{\to} L \iff \forall \varphi \in \mathcal{D}(U): (L_k, \varphi) \to (L, \varphi)\).

- **Examples:** \(L^p_{\text{loc}} \subset \mathcal{D}'(U)\). Aside: \(L^p_{\text{loc}} \subset L^q_{\text{loc}}\) for \(p > q\), (not for \(L^p\)). **Radon measure** (A Borel measure that is finite on compact sets), \(\delta\) function, Cauchy Principal value.

- **Derivative:** \((\partial^\alpha L, \varphi) = (-1)^{|\alpha|}(L, \partial^\alpha \varphi)\).

- **Differentiation is continuous.**

- **Partial differential operator:** \(P = \sum_{|\alpha| \leq N} c_\alpha(x) \partial^\alpha\), adjoint, fundamental solution: \(PK = \delta\).

- **Schwartz class:** \(\mathcal{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)\)

\[
\| \varphi \|_{\alpha, \beta} := \sup_x |x^\alpha \partial^\beta \varphi(x)| < \infty \quad \forall \alpha, \beta.
\]

A polynormed, metrizable space (Use \(\sum 2^{-k} \sum_{|\alpha| + |\beta| = k} \| \cdot \|_{\alpha, \beta} \)). Complete, too. (Arzelà-Ascoli).

- **Examples:**
  - \(\mathcal{D} \subset \mathcal{S}\) (convergence carries over, too.)
  - \(\exp(-|x|^2) \in \mathcal{S}\), but not \(\in \mathcal{D}\).

- **Fourier Transform:**

\[
\hat{\varphi}(\xi) = \mathcal{F}_\varphi(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} \varphi(x) dx
\]

- Basic estimates:

\[
\| \hat{\varphi}(\xi) \|_{L^\infty} \leq C \| (1 + |x|)^{-n+1} \varphi(x) \|_{L^\infty} \leq C \| \varphi \|_{L^1} < \infty,
\]

\[
\| \partial^\beta_x \hat{\varphi}(\xi) \|_{L^\infty} \leq C \| (1 + |x|)^{-n+1} x^\beta \varphi \|_{L^\infty}
\]

\[
\| \xi^\alpha \hat{\varphi}(\xi) \|_{L^\infty} \leq C \| (1 + |x|)^{-n+1} \partial^\alpha_x \varphi \|_{L^\infty}
\]

\[
\| \hat{\varphi} \|_{\alpha, \beta} \leq C \| (1 + |x|)^{-n+1} x^\beta \partial^\alpha_x \varphi \|_{L^\infty} \quad \Rightarrow \quad \hat{\varphi} \in C^\infty.
\]

- **Dilation:** \(\sigma_\lambda \varphi(x) = \varphi(x/\lambda)\). \((\mathcal{F}_\sigma \varphi) = \lambda^\alpha \sigma_{1/\lambda} \mathcal{F}_\varphi\).
6.1 Tempered Distributions

- Translation: \( \tau_h \varphi (x) = \varphi (x - h) \). \( (\mathcal{F} \tau_h \varphi) = e^{-i k \cdot \xi} \mathcal{F} \varphi \).

- Inversion formula:
  \[
  \varphi (x) = \frac{1}{(2 \pi)^{n/2}} \int_{\mathbb{R}^n} e^{i x \cdot \xi} \hat{\varphi} (\xi) \, d\xi = \mathcal{F}^* \varphi = \mathcal{F} \mathcal{R} \varphi ,
  \]
  where \( \mathcal{R} \varphi (x) = \varphi (-x) \).

\( \mathcal{F} \) is an isomorphism of \( \mathcal{S} \), with \( \mathcal{F} \mathcal{F}^* = \text{Id.} \)

Proof: \( (\mathcal{F} \mathcal{F}^* - \text{Id}) e^{-i |x|^2} = 0 \), then for dilations and translations, linear comb. of which are dense in \( \mathcal{S} \). \( \mathcal{F} \) is 1-1, \( \mathcal{F}^* \) is onto, but \( \mathcal{F}^* = \mathcal{R} \mathcal{F} \).

- \( \mathcal{F} \) isometry of \( L^2 \), \( \mathcal{F} \) continuous from \( L^p \) to \( L^q \), where
  \[
  \frac{1}{p} + \frac{1}{q} = 1, \quad p \in [1, 2].
  \]

In particular \( p = 1, \quad q = \infty \).

Proof: Show \( \mathcal{S} \) dense in \( L^p \) (see below), extend \( \mathcal{F} \), use Plancherel for \( L^2 \).

- Mollifier: \( \eta \in C_0^\infty \), \( \int \eta = 1 \). \( \eta_N (x) := N^n \eta (N x) \).

- \( C_0^\infty (\mathbb{R}^n) \) is dense in \( L^p (\mathbb{R}^n) \). \( 1 \leq p < \infty \)

Proof: \( \| \eta_N * f - f \|_{L^p} \to 0 \) holds for step functions. Step functions are dense in \( L^p (\mathbb{R}^n) \).

\[
\| f * \eta_N \|_{L^p} \leq C \| f \|_{L^p} \quad \text{(Young’s)}
\]

Pick \( g \) a step function such that \( \| f - g \|_{L^p} < \varepsilon \). Now measure

\[
\| f * \eta_N - f \|_{L^p} = \| f * \eta_N - g * \eta_N + g * \eta_N - g + g - f \|_{L^p}.
\]

- \( C_0^\infty (\mathbb{R}^n) \) is dense in \( \mathcal{S} \).

Proof: Smooth cutoff.

- Plancherel’s Theorem: \( (\mathcal{F} f, \mathcal{F} g)_{L^2} = (f, g)_{L^2} \).

Proof: by Fubini.

\( \mathcal{F} : L^1 (\mathbb{R}^n) \to \hat{\mathcal{C}} (\mathbb{R}^n) \), with \( \hat{\mathcal{C}} := \{ h : \mathbb{R}^n \to \mathbb{R}; h (x) \to 0 (x \to \infty) \} \).

Proof: \( \mathcal{S} \) is dense in \( L^1 \). Well-defined: Take \( \varphi_k, \psi_k \to f \in L^1 \), show \( \mathcal{F} \varphi_k - \mathcal{F} \psi_k \to 0 \) in \( L^\infty \).

Goes to \( \hat{\mathcal{C}} : \) unproven.

- Linear operator of type \( (r, s) \):
  \[
  \| K \varphi \|_{L^r} \leq C(r, s) \| \varphi \|_{L^s}.
  \]

\( \mathcal{F} \) is of type \((1, \infty)\) and \((2, 2)\).

- Riesz-Thorin Convexity Theorem: \( \mathcal{F} \) of type \((r_0, s_0)\) and \((r_1, s_1)\)

  \[
  \frac{1}{r} = \frac{\theta}{r_0} + \frac{1 - \theta}{r_1}, \quad \frac{1}{s} = \frac{\theta}{s_0} + \frac{1 - \theta}{s_1}
  \]

Then \( \mathcal{F} \) of type \((r, s)\) for \( \theta \in [0, 1] \).
Proof: 1. $\gamma$ maps to $\mathbb{R}$. 2. $\gamma$ sequentially continuous. 3. $\gamma \in C^1$ (FD). 4. $\gamma \in C^\infty$ (induction). 5. $(\eta * L, \varphi) = (\gamma, \varphi)$ (Riemann sums).

- $\mathcal{D}$ is dense in $\mathcal{D}'$.
  Proof: $\chi_m := 1_{[-m,m]}$. Fix $L \in \mathcal{D}'$, $L_m := \chi_m(\eta_m * L) \in D \to L$ in $\mathcal{D}'$.

- $\mathcal{S}$ is dense in $\mathcal{S}'$.
  (because $\mathcal{D}$ is already dense in $\mathcal{D}'$.)

- Transpose $K^t; \mathcal{S} \to \mathcal{S}$ for $K; \mathcal{S} \to \mathcal{S}$ as by $(K^t L, \varphi) := (L, K\varphi)$.

- $\mathcal{F}; \mathcal{S}' \to \mathcal{S}'$ continuous.

- $\mathcal{F}\delta = 1/(2\pi)^{n/2}$.

- $0 < \beta < n, \quad C_\beta = \Gamma((n - \beta)/2)$

- $\mathcal{F}(C_\beta |x|^{-\beta}) = C_{n-\beta}|x|^{-(n-\beta)}$.

Use this to solve Laplace’s equation.

7 Hyperbolic Equations

- General constant coefficient problem. $P(D, \tau) = \tau^m + \tau^{m-1}P_1(D) + \ldots + P_m(D)$

- Duhamel’s principle: Solve $P(D, \tau)u = f$ by solving the standard problem $P(D, \tau)u_s = 0$, $u_s(0) = 0$, $\partial_t^{m-1}u_s(0) = g$ and finding

$$u(x, t) = \int_0^t u_s ds.$$ 

- Treat remaining ICs by solving standard problems for $\tau^{m-1}P_1, \ldots, \tau^0P_m$, each time adding to the right hand side, which can finally be killed with the above approach.

- Fourier-transforms to $P(i\xi, \tau)\hat{u} = 0$, with $\tau = \partial_t$.

- Initial conditions $\tau^{0\ldots m-2}\hat{u}(\xi, 0)$, $\tau^{m-1}\hat{u}(\xi, 0)$.

- Representation of the solution:

$$Z(\xi, t) = \frac{1}{2\pi} \int_\Gamma \frac{e^{i\lambda t}}{P(i\xi, i\lambda)} d\lambda$$

$$P(i\xi, \tau)Z = \frac{1}{2\pi} \int_\Gamma P(i\xi, i\lambda) \frac{e^{i\lambda t}}{P(i\xi, i\lambda)} d\lambda = \frac{1}{2\pi} \int_\Gamma e^{i\lambda t} d\lambda = 0,$$

where $\Gamma$ is a path around the roots.

- Classical solution requires $u \in C^m$. Requires $\forall T \exists C_T, N$:

$$|\tau^k Z(\xi, t)| \leq C_T (1 + |\xi|)^N.$$ 

- Hyperbolicity: A standard problem is hyperbolic: $\Leftrightarrow \exists a C^m$ solution for all $g \in S(\mathbb{R}^n)$.

- Gårding’s Criterion: It’s hyperbolic iff $\exists c \in \mathbb{R} : P(i\xi, i\lambda) \neq 0$ for all $\xi$ and $\text{Im} \lambda \leq -c$.

- Paley-Wiener Theorem: $g \in L^1 \Rightarrow \hat{g}$ entire.

8 Conservation Laws

- $u_t + f(u)_x = 0$. 

Why are they called called conservation laws?
\[
\frac{d}{dt} \int u = \int u_t = \int f(u)_x = f(b) - f(a) \to 0.
\]

- **Inviscid Burgers’ Equation**: \( u_t + (u^2)_x = 0 \).
- **Characteristics**: Assume \( u = u(x(t), t) \),
  \[
  \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial t}
  \]
  Compare shape with
  \[
  0 = u_x f'(u) + u_t,
  \]
  obtain \( dx/dt = f'(u) \).
- **Weak solution**: slap test function onto equation, integrate by parts.
- **Rankine-Hugoniot**:
  \[
  \text{shock speed} = \frac{[f(u)]}{[u]}
  \]
  Apply weak solution formula across a jump, consider normal geometrically to obtain speed.
- **Riemann problem**: Jump IC. \( \to \) non-uniqueness of the weak solution for jump up: rarefaction wave or shock with correct speed?
- **Hopf’s treatment of Burger’s Equation**:
  - Add viscosity to get \( u_t + (u^2/2)_x = \varepsilon u_{xx} \).
  - Put \( U \) as an antiderivative of \( u \).
  - Gives Hamilton-Jacobi PDE \( U_t + U_x^2/2 = \varepsilon U_{xx} \).
  - Now try to rewrite that into a linear equation, by assuming \( \psi = \psi(u) \). Yields ODE \( C\psi'' + C\psi' = 0 \), solution \( \psi = \exp(-U/2\varepsilon) \).
  - This gives the heat equation \( \psi_t = \varepsilon \psi_{xx} \).
  - \[
  u = 2\varepsilon \frac{\psi_x}{\psi} = \frac{\int \frac{x-y}{t} \exp(-G/2\varepsilon)dy}{\int \exp(-G/2\varepsilon)dy} = \frac{x}{t} - \langle y \rangle \frac{x}{t} = \frac{x - \text{argmin } G}{t}
  \]
  with \( G = (x - y)^2/2t + U_0 \).
  - \( a_- = \inf \text{argmin } G, \ a_+ = \sup \text{argmin } G \).
- **Properties**: well-defined, increasing, \( a_+ (\to ) \leq a_- (\to ) \), \( a_- \) left-continuous, \( a_+ \) right-continuous, go to \( \pm \infty \). Equal except for a countable set of shocks.
- **Hopf’s theorem**:
  \[
  \frac{x - a_+}{t} \leq \liminf_{\varepsilon \to 0} u^\varepsilon \leq \limsup_{\varepsilon \to 0} u^\varepsilon \leq \frac{x - a_-}{t}
  \]

- \( u_0 \in \text{BC} \) (bounded, continuous) \( \Rightarrow u(\cdot, t) \in \text{BV}_{\text{loc}} \). **Globally BV?**
  Proof: \( x, a_+, a_- \) are increasing \( \Rightarrow \) differences in \( \text{BV}_{\text{loc}} \).
- Vanishing viscosity solutions are weak solutions.
  Proof: Pass to vanishing viscosity under integral using DCT and boundedness.
- Cole-Hopf solutions produce rarefaction \( x/t \) for jump up, shock for jump down.
- More properties:
  - \( \lim_{\varepsilon \to 0} u^\varepsilon \) exists except for a countable set. \( u = \lim u^\varepsilon \in \text{BV}_{\text{loc}} \) with left and right limits.
    Proof: \( u \) is a difference of increasing functions.
  - **Lax-Oleinik entropy condition**: \( u(x-, t) > u(x+, t) \) at jumps.
“Characteristics never leave a shock.”
Proof: Travelling waves for Burgers with viscosity only exist for \( u_- > u_+ \).

\( \circ \) \( x \) a shock location:
- Shock speed: 
  \[
  \text{shock speed} = \frac{f(u)}{u} = \frac{1}{2}(u_+ + u_-),
  \]
- Shock speed: 
  \[
  \text{shock speed} = \frac{1}{2}(u(x_-, t) + u(x_+, t)) = \int_{a_-}^{a_+} u_0(y) dy,
  \]
- Shock speed: 
  \[
  (a_+ - a_-) \text{ shock speed} = \int_{a_-}^{a_+} u_0(y) dy.
  \]

The last equation here is a momentum conservation equality.
Proof: \( G(a^+) = G(a^-) \).

- **Entropy/entropy-flux pair**: \( \Phi, \Psi : \mathbb{R}^m \to \mathbb{R} \) smooth are an ef/ef pair for \( u_t + f(u)_x = 0 \): \( \Leftrightarrow \Phi \) convex, \( \Phi' f' = \Psi' \). Then \( \Phi(u)_t + \Psi(u)_x = 0 \) for perfectly smooth solutions, otherwise \( \Phi(u)_t + \Psi(u)_x \leq 0 \) in the distributional sense, which means
  \[
  \int_0^\infty \int_{-\infty}^\infty \Phi(u) v_t + \Psi(u) v_x dx dt \geq 0.
  \]
  for smooth non-negative \( v \).

- By the vanishing viscosity method, we get an entropy solution.
  Proof: Multiply the viscosity-added c.law by \( \Phi' \). Use chain rule on \( \Phi(u_\varepsilon')_{xx} \). Use convexity of \( \Phi \) to show one term involving \( \varphi'' \) non-negative. Multiply by a non-negative smooth function, let \( \varepsilon \to 0 \) to obtain entropy inequality.

- **Entropy solution**: \( u \) is an entropy solution of a c.law if \( u \) is a weak solution that satisfies the entropy condition for every ef/ef pair.

- **Dissipation measure**:
  \[
  \frac{d}{dt} \int (u^\varepsilon)^2 = -2\varepsilon \int u_x^2 dx.
  \]
  Assuming a traveling wave solution of the form
  \[
  u^\varepsilon = v \left( \frac{x - ct}{\varepsilon} \right),
  \]
  we find
  \[
  \frac{d}{dt} \int (u^\varepsilon)^2 = \frac{(u_- - u_+)^3}{6}.
  \]

- **Kružkov’s Uniqueness Theorem**: \( L^\infty \) Entropy solutions \( u, v, S_t \) cuts of the event cone (given by max. speed \( c^* = \max_{\text{range } u} |f'| \)). Then for \( t_1 < t_2 \)
  \[
  \int_{S_{t_2}} |u - v| \leq \int_{S_{t_1}} |u - v|.
  \]
  Proof: Doubling trick, clever choice of test functions.
  Implies uniqueness.

### 9 Hamilton-Jacobi Equations

- \( u_t + H(Du, x) = 0 \).
- Example: Curve evolving with normal velocity: \( u_t + \sqrt{1 + |D_x u|^2} = 0 \).
- Non-Example: Motion by mean curvature \( u_t = u_{xx}/(1 + u_x)^2 \) (parabolic).
- Example: Substitute \( U = \int u \) in conservation laws.
• PDE is infinitely-many-particle limit of Hamilton ODE

\[
\begin{align*}
\dot{x} &= \partial_p H(p, x) \\
\dot{p} &= -\partial_x H(p, x),
\end{align*}
\]

which coincides with characteristic equation of PDE.

• Mechanics motivation:
  - \(L(q, x) = T - V\)
  - Lagrange’s Equation

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial q} \right) = \frac{\partial L}{\partial x}.
\]

Way to see this: If RHS = 0, then \(L\) symmetric in \(x\), so LHS becomes conserved. (Noether’s theorem.) Equivalent to Hamilton’s ODE (Proof: \(H = \max_q (qp - L(x, q, t))\), where \(q = q(x, p, t)\) is the solution of \(p = \partial_q L(v)\).

- Action, given path \(x(t)\):
  \[S(x) = \int_0^t L(\dot{x}, x, t) dt\]

- Principle of least action: \(\min S \Leftrightarrow\) Lagrange’s Equation.
  Proof: \(u + \varepsilon v\), derivative by \(\varepsilon\), the usual.

- Generalized momentum: \(p = \partial_q L\). Assumed solvable for \(q\).

- Hamiltonian: \(H = T + V = p \cdot q - L = 2T - (T - V) = T + V\).

- Legendre transform: More general way of obtaining \(H\). Assume \(L(q)\) (dropping dependencies!) convex, \(\lim_{|q| \to \infty} L(q)/|q| = \infty\). Then

\[
H(p) = L^*((p) = \sup_{q} \{ p \cdot q - L(q) \}.
\]

Solved when \(p = \partial_q L\), but in a more general sense.

Duality: Edge \(\leftrightarrow\) Corner. Subdifferentials.

- \(L\) convex \(\Rightarrow L^{**} = L\).
  Proof: Prove convexity and superlinearity of \(L^*\). Use symmetry

\[
H(p) + L(q) \geq p \cdot q
\]

to prove two sides of the equality \(H^* = L\).

• Hopf-Lax formula: \(g\) is IC

\[
u(x, t) = \inf \left\{ \int L(\dot{x}) dx + g(y), x(0) = y, x(t) = x \right\} = \min \left\{ t L(\frac{x - y}{t}) + g(y) \right\}.
\]

Proof: Inf bounded above by straight-line characteristic. Lower bound works by Jensen’s inequality.

• Semigroup Property.
  Proof: Always pick particular solutions, prove both sides of the inequality.

• \(u\) defined by Hopf-Lax is Lipschitz if \(g\) is Lipschitz.
  Proof: Lipschitzicity for given \(t\) is immediate (pick good \(z\)). Transform problem to comparison with \(t = 0\) by semigroup property. Temporal estimate is screwy, involves special choices in inf.

• \(u\) by Hopf-Lax is differentiable a.e. and satisfies the H-J PDE where it is.
  Proof: Rademacher’s Theorem. Prove \(u_t + H(Du) \leq 0\) for forward in time by taking increments \(\to 0\), using inequality with Legendre transform.
Lipschitz+Differentiable solution a.e. is not sufficient for uniqueness. (45-degree angle trough vs. 90-degree trough)

\( f: \mathbb{R}^n \to \mathbb{R} \) semiconcave if

\[ f(x + z) - 2f(x) + f(x - z) \leq C|z|^2 \]

for some \( z \).

\( \Leftrightarrow f(z) - C/2|z|^2 \) is concave.

\( \Leftrightarrow \) “can be forced into concavity by subtracting a parabola.”

\( \Leftrightarrow C^2 \) and bounded second derivatives implies semiconcavity.

\( g \) semiconcave \( \Rightarrow u \) semiconcave.

Clever choice of test locations in Hopf-Lax.

\( H: \mathbb{R}^n \to \mathbb{R} \) uniformly convex: \( \Leftrightarrow \)

\[ \sum_{i,j} H_{p,p_i} \xi_i \xi_j \geq j|\xi|^2. \]

If \( H \) uniformly convex. Then \( u \) is semiconcave (indep. of initial data)

Proof: Taylor, mess about with Hopf-Lax.

Now \( H(p) \to H(p, x) \) nonconvex.

Vanishing Viscosity Method: Use \( u_t + H(Du, x) = \varepsilon \Delta u \). Locally uniform convergence follows from Arzelà-Ascoli.

\( u \) is a viscosity solution: \( \Leftrightarrow u = g \) on \( \mathbb{R}^n \times \{t = 0\} \), for each \( v \in C^\infty(\mathbb{R}^n \times (0, \infty)) \)

\( u - v \) has a local maximum at \((x_0, t_0) \Rightarrow v_t(x_0, t_0) + H(Dv(x_0, t_0)) \leq 0 \) (and \( \min \to \geq \)).

If \( u \) is a vanishing viscosity solution, then it is a viscosity solution.

Proof: Convergence is locally uniform as \( \varepsilon \to 0 \). Thus for each fixed ball around a local strict maximum in \( u - v \), a local maximum in \( u^\varepsilon - v \) exists if \( \varepsilon \) is small enough. There, \( v_x = u_x^\varepsilon \) and \( v_t = u_t^\varepsilon \) and \( -\Delta u^\varepsilon \geq -\Delta v \). \( v_t + H(Dv) \leq 0 \) follows. Generalize to non-strict maxima by adding parabolas.

A classical solution of a H-J PDE is a viscosity solution.

Proof: Maximum of \( u - v \to \) derivatives are equal \( \Rightarrow \) PDE.

Touching by \( C^1 \) function: \( u \) continuous. \( u \) differentiable at \( x_0 \).

Then \( \exists v \in C^1: v(x_0) = u(x_0) \), \( u - v \)

has a strict local max.

\( u \) viscosity solution \( \Rightarrow u \) satisfies H-J wherever it is differentiable

Proof: Mollify touching function, \( u - v^\varepsilon \) maintains strict max., verify definition of Viscosity solution. (Mollification necessary because test functions are required to be \( C^\infty \).)

Uniqueness: \( H \in \text{Lip}_p(C) \cap \text{Lip}_x(C^1 + |p|) \Rightarrow \) uniqueness.

Proof: doubling trick again.

### 10 Sobolev Spaces

\( 1 \leq p < \infty. \)

- \( \|u\|_{k,p;\Omega} \equiv \sum_{|\alpha| \leq k} \|D^\alpha u\|_{p,\Omega} \)
- \( W^{k,p}(\Omega) := \{ u \in D'(\Omega): D^\alpha u \in L^p(\Omega), |\alpha| \leq k \} \) Banach space.
- \( W^{k,p}_0(\Omega) := \text{cl}(D(\Omega), \|\cdot\|_{k,p;\Omega}) \).
- \( u \in W^{l,p}(\Omega) \). \( \Omega' \subset \subset \Omega \) open \( \Rightarrow \exists u_k \in C^\infty(\Omega'): \|u_k - u\|_{l,p;\Omega} \to 0. \)
  
  Proof: Mollification, throw derivatives onto \( u \) by integration by parts.
- \( u \in W^{l,p}(\Omega) \). \( \Omega \) bounded \( \Rightarrow \exists u_k \in C^\infty(\Omega) \cap W^{l,p}(\Omega): \|u_k - u\|_{l,p;\Omega} \to 0. \)
Proof: Exhaust $\Omega$ by $U_k := \{\text{dist}(x, \partial U) > 1/k\}$. Consider smooth partition of unity $\zeta_i$ subordinate to $V_i := \Omega_{i+3} \setminus \Omega_{i+1}$. $u_i := \eta \cdot (\zeta_i u)$ s.t. $\| u_i - \zeta_i u \|_{l,p} < \delta 2^{-i-1}$. Give one more set of wiggle room on each side for mollification. $v := \sum \zeta_i u_i \in C^\infty$ because there’s only a finite number of terms for fixed point/set. Then estimate $\| u - v \|_{l,p}$.

- Typical idea: Consider
  \[ f^*(x) = \lim_{r \to 0} \int_{B(x,r)} f(y)dy. \]
  \[ u \in W^{1,p}(\Omega), \Omega' \subset \subset \Omega. \]

  - There exists a representative on $\Omega'$ that is absolutely continuous on a line and whose classical derivative agrees a.e. with the weak one.

  - If the above is true of a function, then $u \in W^{1,p}(\Omega)$.

Proof: WLOG $p = 1$ (Jensen). WTF?

Consequences: $W^{1,p}$ closed wrt. max, min, abs. value, $\cdot^+$. $\Omega$ connected, $Du = 0 \Rightarrow u$ constant.

10.1 Campanato

- Oscillation:
  \[ \text{osc}_U = \sup_{x,y \in U} |u(x) - u(y)|. \]

- $C^{0,\alpha} := \{ |u(x) - u(y)| \leq C|x - y|^\alpha \}$. $\| u \|_{C^{0,\alpha}} := \| u \|_{C(U)} + \sup_{x \neq y} |u(x) - u(y)|/|x - y|^\alpha$.

- $C^{k,\alpha} := D^\alpha \subset C^{0,\alpha}$. Norm: sum over multi-indices.

- Campanato’s Inequality: $u \in L^1_{\text{loc}}(\Omega), 0 < \alpha \leq 1, \exists M > 0$:
  \[ \int_B |u(x) - \bar{u}_B(x)|dx \leq Mr^\alpha. \]

Then $u \in C^{0,\alpha}(\Omega)$ and $\text{osc}_{B(x,r/2)} u \leq CMr^\alpha$. $\bar{u}_B$ is the mean over $B$.

Proof: $x$ a Lebesgue point of $u$, $B(x, r/2) \subset B(z, r)$. Then $|\bar{u}_{B(x,r/2)} - \bar{u}_{B(z,r)}| \leq 2^n Mr^\alpha$. Iteration via geometric series and Lebesgue-pointy-ness yields

\[ |u(x) - \bar{u}_{B(z,r)}| \leq C(n, \alpha)M r^\alpha. \]

For two Lebesgue points,

\[ |u(x) - u(y)| \leq |u(x) - \bar{u}_{B(z,r)}| + |\bar{u}_{B(z,r)} - u(y)| \leq C(n, \alpha)M r^\alpha. \]

10.2 Sobolev

- Gagliardo-Nirenberg-Sobolev: $u \in C^1_c(\mathbb{R}^n)$, $1 \leq p < n \Rightarrow$

  \[ \| u \|_{p^*} \leq C \| Du \|_p, \]

  where

  \[ \frac{1}{p^*} + \frac{1}{n} = \frac{1}{p} \Rightarrow p^* > p. \]

- Considering what happens when you scale functions $u \to u_\lambda(x) := u(\lambda x)$, these exponents are the only ones possible.

- If we choose $p = 1$, then the best constant comes to light by choosing $u = 1_{B(0,1)}$, giving the isoperimetric inequality.

Proof: Suppose $p = 1$ at first. Compact support $\Rightarrow$

\[ u(x) \leq \int_{-\infty}^{\infty} |Du(x, y_1, x, ..., x)|dy_i \quad (i = 1, ..., n). \]
Then
\[ |u(x)|^{n/(n-1)} \leq \left( \prod_i \int \ldots dy_i \right)^{1/(n-1)}. \]
Integrating this gives
\[ \int |u|^{n/(n-1)}dx_1 \leq \left( \int |Du|dx_1 \right)^{1/(n-1)} \left( \prod_{i=2}^n \int \int |Du|dx_1dy_i \right)^{1/(n-1)} \]
by pulling out an independent part and using generalized Hölder. Then iterate the same trick. To obtain for general \( p \), use on \( v = |u|^\gamma \) with suitable \( \gamma \).

### 10.3 Poincaré and Morrey
- **Riesz potential:** \( 0 < \alpha < n \)
  \[ I_\alpha(x) = |x|^{\alpha - n} \in L^1_{\text{loc}}(\mathbb{R}^n). \]
- \( \|I^* f\|_{L^p} \leq C\|f\|_{L^p} \).
- **Poincaré’s Inequality:** \( \Omega \) convex, \( |\Omega| < \infty, d = \text{diam}(\Omega) \), \( u \in W^{1,p}(\Omega) \). Then
  \[ \left( \int_\Omega |u(x) - \bar{u}_\Omega|^p \right)^{1/p} \leq Cd \left( \int_\Omega |Du|^p \right)^{1/p}. \]
  Proof: Use calculus to derive
  \[ |u(x) - \bar{u}| \leq \frac{d^n}{n} \int_\Omega \frac{|Du(y)|}{|x - y|^{n-1}}dy. \]
  Then use potential estimate.
- **Morrey’s Inequality:** \( u \in W^{1,1}_{\text{loc}}(\Omega), 0 < \alpha \leq 1 \). If \( \exists M > 0 \) with
  \[ \int_{B(x,r)} |Du| \leq Mr^{n-1+\alpha}, \]
  for all \( B(x,r) \subset \Omega \). Then \( u \in C^{0,\alpha}(\Omega) \) and \( \text{osc}_{B(x,r)} u \leq CMr^\alpha \).
- **Morrey=Poincaré+Campanato** in \( W^{1,1} \).
- **More general Morrey:** \( u \in W^{1,p}(\mathbb{R}^n), n < p \leq \infty \). Then \( u \in C^{0,1-n/p}(\mathbb{R}^n) \) and
  \[ \text{osc}_{B(x,r)} u \leq r^{1-n/p} \|Du\|_{L^p}. \]
  If \( p = \infty \), \( u \) is locally Lipschitz.
  Proof: Use Jensen \( (\cdot)^p \gamma \) on Poincaré’s RHS. Then apply Campanato.

### 10.4 BMO
- **BMO seminorm:**
  \[ [u]_{\text{BMO}} := \sup_B \int_B |u - \bar{u}_B|dx \]
  \[ \text{BMO} := \{ [u]_{\text{BMO}} < \infty \}. \]
- **John-Nirenberg:** \( W^{1,n}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \subset \text{BMO}(\mathbb{R}^n) \).
  Proof: Poincaré-then-Jensen.
- For a compact domain, \( L^p \subset L^\infty \subset \text{BMO} \).

### 10.5 Imbeddings
- **Imbedding** \( B_1 \rightarrow B_2 \): \( \exists \) continuous, linear, injective map.
Scalar Elliptic Equations

- \( W^{1,p}(\mathbb{R}^n) \to L^p \) for \( 1 \leq p < n \) (Sobolev inequality)
- \( W^{1,p}(\mathbb{R}^n) \to \text{BMO} \) for \( p = n \)
- \( W^{1,p}(\mathbb{R}^n) \to C^{0,1-n/p}_{\text{loc}} \) (Morrey)

\( \Omega \) bounded now.

- \( W^{1,p}(\Omega) \to L^q(\Omega) \) for \( 1 < p < n \) and \( 1 \leq q < p^* \). Proof: Hölder-then-Sobolev:
  \[
  \|u\|_{L^q} \leq \|u\|_{L^{p^*}}|\Omega|^{1-q/p^*} \leq \|Du\|_{W^{1,p}}.
  \]

- \( W_0^{1,p}(\Omega) \to C^{0,1-n/p}(\Omega) \) for \( n < p \leq \infty \).
- Compact imbedding \( B_1 \hookrightarrow B_2 \): The image of every bounded set in \( B_1 \) is precompact in \( B_2 \).
  (precompact: has compact closure)
- Rellich-Kondrachov:
  - \( W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \) for \( 1 < p < n \) and \( 1 \leq q < p^* \).
    In Evans, we need \( \partial U \subset C^1 \). Our notes do not.
    
    Proof:
    - Grab a \( W^{1,p} \)-bounded sequence \( u_m \).
    - Mollify it to \( u_m^\epsilon \)
    - Use an \( \epsilon \)-derivative trick to show \( \|u_m^\epsilon - u_m\|_{L^1} \leq \epsilon \|Du_m\|_{L^p} \to 0 \)
    - Interpolation inequality for \( L^p \): \( \|u_m^\epsilon - u_m\|_{L^1} \leq \|u_m^\epsilon - u_m\|_{L^{1/p}}\|u_m^\epsilon - u_m\|_{L^{p'}} \to 0 \), also
      using GNS.
    - For fixed \( \epsilon \), \( u_m^\epsilon \) is bounded and equicontinuous (directly mess with convolution).
    - Use Arzelà-Ascoli and a diagonal argument to finish off.
  - \( W_0^{1,p}(\Omega) \hookrightarrow C^0(\bar{\Omega}) \subset L^p(\Omega) \) for \( n < p \leq \infty \).
    Proof: Morrey’s Inequality, then Arzelà-Ascoli.

11 Scalar Elliptic Equations

- \( Lu = \text{div}(A Du + bu) + c \cdot Du + d u \).
- Motivation: Calculus of Variations.
- Weak Formulation: \( u \in W^{1,2}(\Omega), v \in C^1_c(\Omega) \)
  \[
  B[u,v] := \int_\Omega (Dv^T A Du + b \cdot Dv u) - (c \cdot Du + d u) v \, dx.
  \]
- Generalized Dirichlet Problem: \( Lu = g + \text{div} \, f \) on \( \Omega \), \( u = \varphi \) on \( \partial \Omega \), i.e. \( B[u,v] = F(v) \) with
  \[
  F(v) := \int_\Omega Dv \cdot f - g v \, dx.
  \]
- Assumptions:
  - \( (E_1) \). Strict ellipticity: \( \exists \lambda > 0: \xi^T A \xi \geq \lambda |\xi|^2 \)
  - \( (E_2) \). Boundedness: \( A, b, c, d \in L^\infty \), i.e. \( \|\text{Tr}(A^T A)\|_{L^\infty} \leq \Lambda^2, \frac{1}{\lambda^2} (\|b\|_{\infty} + \|c\|_{\infty}) + \frac{1}{\lambda} (\|d\|_{\infty}) \leq \nu \).
  - \( (E_3) \). \( \text{div} \, b + d \leq 0 \) weakly, i.e.
    \[
    \int_\Omega d v - b \cdot Dv \, dx \leq 0
    \]
    for \( v \in C^1_c(\Omega), v \geq 0 \).
11. Existence Theory

11.1 Existence Theory

- "≤" on the boundary: \( u \leq v \Leftrightarrow (u - v) \leq 0 \). \( (u - v)^+ \in W^{1,2}_0(\Omega) \).
- "sup" on the boundary: \( \sup_{\partial \Omega} u = \inf \{ k \in \mathbb{R} : u \leq k \text{ on } \partial \Omega \} \).
- \( u \) is a subsolution: \( \Leftrightarrow B[u, v] \leq F(v) \Leftrightarrow Lu \geq g + \div f \).
- **Non-divergence form:**

\[
0 = AD^2U + b \cdot Du + d \cdot u
\]

(Not equivalent!)
- Classical Maximum Principle: Holds if \( d \leq 0 \).
- Weak Maximum Principle: \( Lu \geq 0 \Leftrightarrow B[u, v] \leq 0 \) for \( v \geq 0 \) and \( (E_1), (E_2), (E_3) \). Then \( \sup_\Omega u \leq \sup_{\partial \Omega} u^+ \).

Proof:

\( \circ \) Use \( B[u, v] \leq 0 \) for \( v \geq 0 \) and \( (E_3) \) to establish

\[
\int Du^TADv - (b + c)Du \cdot v \leq \int d(u v) - b \cdot D(u v) \leq 0.
\]

Note that \( u v \) is the new test function in \( (E_3) \). Consequently

\[
\int Du^TADv \leq \int (b + c)Du \cdot v.
\]

\( \circ \) Suppose \( l = \sup_{\partial \Omega} u \leq k < \sup_\Omega u \). Set \( \Gamma := \{ u > k \} \) and achieve a \( \| Dv \|_{L^2} \leq C \| v \|_{L^2} \) estimate by using ellipticity, the above and boundedness. Use the Sobolev inequality to get \( \| v \|_{L^2} \leq \cdots \leq |\Gamma|^{1/n} \| v \|_{L^2} \), and so \( |\Gamma| > 0 \) independently of \( k \). Let \( k \to \sup_\Omega \) to obtain a contradiction. (Note \( \sup_\Omega < \infty \) because \( u \in W^{1,2}(\Omega) \).)

Remarks:

\( \circ \) Implies uniqueness.
\( \circ \) No assumptions on boundedness, smoothness or connectedness of \( \Omega \).

- Implies uniqueness.

**11.1 Existence Theory**

- **Existence:** \( \Omega \) bounded, \( (E_1), (E_2), (E_3) \). Then \( \exists! \) solution of the generalized Dirichlet problem.

\( \circ \) Reduce BC to \( H^1_0 \) by subtracting arbitrary function and handling RHS.

\( \circ \) Prove coercivity estimate

\[
B[u, u] \geq \frac{\lambda}{2} \int \Omega |Du|^2 dx - \lambda \nu^2 \int \Omega |u|^2 dx.
\]

(Uses: \( (E_1), (E_2), 2a b \leq \lambda a^2 + b^2 / \lambda \).
In Evans, Poincaré enters here. How?)
(For \( \Delta \), Poincaré suffices to show coercivity.)

\( \circ \) \( \text{Id}: W^{1,2}_0 \to (W^{1,2}_0)^* \) is compact.

\[
\text{Id} = \left( \text{Id}^2 \to \mathcal{H}^* \right) \circ \left( \mathcal{H} \to L^2 \right).
\]

\( \circ \) \( L_2 := L - \sigma \text{Id}. \) (\( L \equiv \Delta \) has negative eigenvalues already. But they might be pushed upward by the first- and zeroth-order junk. So we might have to make them even more negative to succeed.)

\( \circ \) \( \to B_{\sigma}[u, v] = B[u, v] + \sigma(u, v)_{L^2}, \) coercivity is maintained.
Lax-Milgram shows existence of inverse $L_{σ}^{-1}$ for the not-so-bad operator $L_{σ}$.

Start with $Lu = g + \text{div } f$, introduce $L_{σ}$, multiply by $L_{σ}^{-1}$ and see what happens.

Weak maximum principle provides uniqueness for $L$, so that the Fredholm alternative provides existence when combined with Rellich.

11.2 Regularity

• Assumptions:
  - $(R_1)$: $(E_1), (E_2)$.
  - $(R_2)$: $f \in L^q(Ω), g \in L^{n/2}, q > n$.

• $(R_1)$, $Lu = g$. $A, b$ Lipschitz. Then for $Ω' \subset⊂ Ω$ we have
  \[ \|u\|_{W^{2,1}(Ω')} \leq C \left( \|u\|_{W^{1,2}(Ω)} + \|g\|_{L^2(Ω)} \right). \]

Proof:
  - Finite Differences.

11.3 Harnack Inequality Stuff

• (Ladyzhenskaya/Uraltseva): $(R_1), (R_2)$. $u \in W^{1,2}$ a subsolution, $u \leq 0$ on $∂Ω$. Then:
  \[ \sup_{Ω} u \leq C \left( \|u^+\|_{L^2(Ω)} + k \right), \]
  where
  \[ k = \frac{1}{\lambda} \left( \|f\|_{L^q} + \|g\|_{L^{n/2}} \right). \]

Proof:
  - Local Boundedness: $(R_1), (R_2)$. $u \in W^{1,2}$ a subsolution. Then:
    \[ \sup_{B(y,R)} u \leq C \left( R^{-n/p} \|u^+\|_{L^2(Ω)} + k(R) \right), \]
    where
    \[ k(R) = \frac{R^{1-n/q}}{\lambda} \left( \|f\|_{L^q} + R^{1-n/q} \|g\|_{L^{n/2}} \right). \]

• Weak Harnack Inequality: $(R_1), (R_2)$, $u \in W^{1,2}(Ω)$ a supersolution and $u \geq 0$ in $B(y,4R) \subset⊂ Ω$. Then
  \[ R^{-n/p} \|u\|_{L^p(B(2R))} \leq C \left( \inf_{y \in B(y,R)} u + k(R) \right). \]

• Strong Harnack Inequality: $(R_1), (R_2), u \in W^{1,2}(Ω)$ a solution with $u \geq 0$. Then
  \[ \sup_{B(y,R)} u \leq C \left( \inf_{B(y,R)} u + k(R) \right). \]

• Strong Maximum Principle: $(R_1), (R_2), (E_3), Ω$ connected, $u \in W^{1,2}$ a subsolution $Lu \geq 0$. If
  \[ \sup_{Ω} u = \sup_{Ω} \]
  then $u = \text{const.}$

Proof: Weak Harnack shows $\{ u = \sup_{Ω} u \}$ is open. $\{ u = \sup_{Ω} u \}$ is relatively closed in $Ω$. Therefore
$\{ u = \sup_{Ω} u \} = Ω$.

Why is $L \text{ const} = 0$?
How dow we know the “relatively closed” part?
• DeGiorgi/Nash: \((R_1), (R_2), u \in W^{1,2}\) solution of \(Lu = g + \text{div } f\). Then \(f\) is locally Hölder and

\[
\text{osc}_{B(y,R)} u \leq CR^\alpha \left( R_0^{-\alpha} \sup_{B(y,R_0)} |u| + k \right)
\]

if \(0 < R \leq R_0\).

Proof:

\[
\text{wlsc because } \parallel u \parallel_{W^{1,2}} \leq \parallel u \parallel_{L^2,\Omega} \quad \text{bounded below by}\]

Looking for \(\inf_{u \in A} I[u]\), where \(A = W^{1,2}_0(\Omega)\).

• Example: Dirichlet’s Principle: \(\Omega\) open, bounded

\[
I[u] = \int_{\Omega} F(Du(x), u)dx.
\]

Looking for \(\inf_{u \in A} I[u]\), where \(A = W^{1,2}_0(\Omega)\).

12 Calculus of Variations

\(\Omega\) open, bounded.

• Idea: solution \(u\), smooth variation \(\varphi\), functional \(I. \partial_x I(\text{u} + \varepsilon \varphi)|_{\varepsilon = 0} = 0\). Integrate by parts, \(\varphi\) was arbitrary \(\rightarrow\) PDE.

• \(u: \Omega \to \mathbb{R}^m\) deformation, \(Du: \Omega \to \mathbb{R}^m \times n\), \(F: \mathbb{R}^m \times n[ \times \mathbb{R}^n] \to \mathbb{R}\).

\[
I[u] = \int_{\Omega} F(Du(x), u)dx.
\]

Looking for \(\inf_{u \in A} I[u]\), where \(A = W^{1,2}_0(\Omega)\).

• Weak lower semicontinuity: \(u_k \rightharpoonup u \Rightarrow I[u] \leq \liminf_{k \to \infty} I[u_k]\).

• \(F\) convex \(\Rightarrow\) \(I\) wloc in \(W^{1,p}_0(\Omega)\).

Proof: Use representation of convex \(F\) as limit of increasing sequence \(\{F_N\}\) of piecewise affine functions. Implies \(\int F_N(Du_k) \rightharpoonup \int F_N(Du)\) (weak convergence \(\lor\) linear/affine functions). Then

\[
\int F_N(Du_k) \leq \liminf_{k \to \infty} \int F(Du_k) = \liminf_{k \to \infty} I[u_k]
\]

and MCT.

• Jensen: \(F(w \star \lim g_k) \leq w \star \lim F(g_k)\).

• Euler-Lagrange Equation: Weak form obtained from \(i(\tau) = I[u + \tau v]\), where \(u = \arg\min I[u]\) and looking at \(i'(0) = 0\).

\[
- \text{div}(F_p(Du)) + F_u(Du, u) = 0.
\]

Also \(i''(0) \geq 0\).

• Motivation for Convexity: \(\rho(s) = 0\)-1 sawtooth. \(\rho' = 1\) a.e.. \(v_\varepsilon(x) = \varepsilon \zeta(x) \rho(x \cdot \varepsilon)\).

\[
\frac{\partial v_\varepsilon}{\partial x_i}(x) \approx \varepsilon \zeta(x) \rho'(x \cdot \varepsilon) \approx \varepsilon \zeta(x) \xi.
\]

Consider \(i''(0) \geq 0 \Rightarrow \xi^T D^2 F \xi \geq 0\) pops out.

• \(m = 1 \Rightarrow (\text{wloc} \iff \text{convexity})\).
Proof: “⇐”: shown above. “⇒”: 2\(^nk\) cube grid on \([0,1]^n\), \(v \in C^\infty_c\).

\[
\begin{align*}
    u_k(x) &= \frac{1}{2k}v(2k(x - \text{cell center})) + z \cdot x.
    \\
    Du_k(x) &= Dv(2k(x - \text{cell center})) + z.
\end{align*}
\]

\(u_k \to z \cdot x, Du_k \rightharpoonup Du\). Then

\[
F(z) \leq \operatorname{wlsc} \liminf_{k \to \infty} \sum_l \int_{Q_l} F(Du_k) = \int_{[0,1]^n} F(z + Dv)
\]

Thus \(I[u]\) has a minimum at the straight line, and for \(i(\tau) = I[u + \tau v]\), \(i'(0) = 0, i''(0) \geq 0\), convexity follows as above.

### 12.1 Quasiconvexity

\(m \geq 2, A = W^{1,p} \cap \{u = g\}_{\partial \Omega}. 1 < p < \infty. F \in C^\infty, F(A) \geq c_1|A|^p - c_2.\)

\[
I[u] = \int_\Omega F(Du(x))dx
\]

- Sawtooth calculation yields rank-one convexity

\[
(\eta \otimes \xi)^T D^2 F(P)(\eta \otimes \xi)
\]

\(\Leftrightarrow F(P + t(\eta \otimes \xi))\) convex in \(t\).

- Quasiconvexity: \(F\) quasiconvex: \(\Leftrightarrow \forall A \in \mathbb{R}^{m \times n}, v \in C^\infty_c([0,1]^n, \mathbb{R}^m):\)

\[
F(A) \leq \int_{[0,1]^n} F(A + Dv)
\]

- If \(|F(A)| \leq C(1 + |A|^p)\), then \(F\) QC \(\Leftrightarrow I\) wlsc.

  - “⇒”: Subdivide domain into cubes,

\[
\int_\Omega F(Du) \approx \int_\Omega F(\text{affine approx to } Du) \leq \int_\Omega F(Du_k) + \text{errors}.
\]

  Use measure theory to keep concentrations (Dirac bumps?) of \(Du\) or \(Du_k\) away from cube boundaries. Mop up the error terms.

  - “⇐”: cubes calculation above.

- Polyconvex: \(F\) is a convex function of minors of \(A\).

- Convex \(\Rightarrow\) PC \(\Rightarrow\) QC \(\Rightarrow\) R1C (converse false).

  Proof of PC \(\Rightarrow\) QC: PC \(\Rightarrow\) wlsc (use convex \(\Rightarrow\) wlsc argument for each minor). wlsc \(\Rightarrow\) QC.

- \(|DF(A)| \leq C(1 + |A|^{p-1})\).

  Proof: Exploit growth estimate above, and QC \(\Rightarrow\) R1C. Use \(f(t) = F(A + t(\eta \otimes \xi))\), which is convex \(\Rightarrow\) locally Lip \(\Rightarrow\) locally \(|f'(0)| \leq \max |f|\).

### 12.2 Null Lagrangians, Determinants

- \(F(Du)\) is a null Lagrangian if E-L

\[
\text{div}(DF(Du)) = \partial x_j(\partial_{A_{ij}} F(Du)) = 0
\]

holds for every \(u \in C^2\).

- \(F\) null Langrangian. Then

\[u = \tilde{u} \text{ on } \partial \Omega \quad \Rightarrow \quad I[u] = I[\tilde{u}]\]

Proof: \(i(\tau) := I[\tau u + (1 - \tau)u], i'(0) = 0\) by E-L.
• Cofactor matrix:
  - $\text{cof}(A_{i,j}) = \det(A_{\setminus i, \setminus j}).$
  - $A^{-1} = \frac{1}{\det A} (\text{cof } A)^T.$
  - $\Rightarrow A^T \text{cof } A = \det A \cdot \text{Id}$
  - $\Rightarrow \partial_{A_{i,j}} \det(A) = (\text{cof } A)_{i,j}$

• $\det(Du)$ is a null Lagrangian, i.e.
  - $\text{div}(\det(Du)) = 0$ for $\det(Du) \neq 0,$ otherwise add $\varepsilon \text{Id}.$

• $u_k \rightarrow u$ in $W^{1,p},$ $n < p < \infty.$ $\Rightarrow \det(Du_k) \rightarrow \det(Du)$ in $L^{p/n}.$ (Morrey/Reshetnyak)
  - Reduce dimension of problem by one by reducing to “does the cofactor matrix converge”?
  - Use $\det(Du) = \text{div} \left( \frac{1}{n} \text{cof}(Du)^T u \right).$
  - Morrey ($n < p$) implies uniform boundedness in $C^{0,1-n/p},$ then use $A-A$ to extract uniformly converging subsequence, settling the deal for the leftover $u$ besides the cofactor matrix.
  - (also holds for $p = n$ if $\det(Du_k) \geq 0$ – no proof.)

• No Retract Theorem: $B = B(0,1).$ There is no continuous map $w: \overline{B} \rightarrow \partial \overline{B}$ with $w(x) = x$ on $\partial B.$
  - Proof: Suppose there is a retract $w.$ By comparison with $\text{Id}$ and identity on the boundary,
    
    $\int \det(Dw) = |B|.$
  
  OTOH, $|w|^2 = 1 \Rightarrow (Dw)^T w = 0 \Rightarrow \det(Dw) = 0.$ Lose smoothness requirement by continuously extending by $\text{Id},$ mollifying and using $B(0,2)$ then.

• Brouwer’s Fixed Point Theorem: $u: \overline{B} \rightarrow \overline{B}$ continuous. $\exists x \in \overline{B}: u(x) = x.$
  - Proof: Assume no fixed point. $w: B \rightarrow \partial B$ is the point on $\partial B$ hit by the ray from $u(x)$ to $x.$ $w$ is a retract because $w$ hits $\partial B$ in $x$ if $x \in \partial B.$ $w$ is continuous.

• Degree of a map: $u \in W^{1,1}$

  $\deg(u) = \int_B \det(Du).$

  Definable for continuous functions by approximation. Is an integer.

13 Navier-Stokes Equations

$G$ open, $\hat{G} := G \times (0, \infty)$ space-time.

• Navier-Stokes Equations:

  $Du/Dt = (\nu \Delta u - \nabla p) + f,$

  $(*)$ is the material derivative $D^* / Dt = \partial_t^* + u \cdot \nabla^*.$

  $\nu = 0 \Rightarrow$ Euler equation. $\nu \neq 0 \Rightarrow$ may as well assume $\nu = 1.$

  • Conservation of mass: $\partial_t \rho + \text{div}(\rho u) = 0.$ Assume $D\rho / Dt = 0 \Rightarrow \nabla \cdot u = 0.$

  • Pressures in a smooth incompressible flow are superharmonic: Take div of NSE.

  • Steady flows: $u \cdot \nabla u + \nabla p = \nu \Delta u.$

  • Bernoulli’s Theorem: ideal ($\nu = 0$), steady flow $u \cdot \nabla u + \nabla p = 0 \Rightarrow \nabla (u^2/2 + p) = 0$
\( \Rightarrow u^2/2 + p = \text{const} \) (still need conservation of mass \( \nabla \cdot u = 0 \))

- **Vorticity**: \( \omega = \text{curl} \ u \)
  \[
  \partial_t \omega + \nabla \times (u \cdot \nabla u) = \Delta \omega, \\
  \nabla \cdot u = 0, \\
  \nabla \times u = \omega.
  \]

In 2D, \( \nabla \times (u \cdot \nabla u) \) becomes \( u \cdot \nabla u \).

- **Helmholtz Projection**: \( P = L^2 \)-closure \( \{ \nabla \varphi : \varphi \in C_c^\infty(\mathbb{R}^n) \} \). \( P^\perp \) (note \( P \) closed) is divergence-free.

\[
L^2 = P \oplus P^\perp
\]

Example: Divergence-free field from sem. 1 final: (continuous) boundary-normal field matters, (discontinuous) tangential field does not.

- **Weak formulation**:
  - Take \( a \in C_c^\infty(\tilde{G}, \mathbb{R}^n) \) div-free, dot NSE with it,
  - i. by parts second term, popping the derivative onto \( a \cdot u \)-product, pull apart, one term is zero,
  - \( \int a \cdot \nabla p = - \int (\text{div} \ a)p = 0 \)
  gives

\[
(W1) - \int_{\tilde{G}} \partial_t a \cdot u + \nabla a \cdot (u \otimes u) + \Delta a \cdot u \, dx \, dt = 0
\]

\[
(W2) \int_{\tilde{G}} \nabla \varphi \cdot u = 0 \quad (\varphi \in C_c^\infty(\mathbb{R}^n))
\]

(where \( A \cdot B = \text{tr}(A^T B) \))

- \( V := \| \cdot \|_V \)-closure \( \{ a \in C_c^\infty(\tilde{G}, \mathbb{R}^n), \nabla \cdot a = 0 \} \)

\[
\| a \|_V := \int_{\tilde{G}} |a|^2 + |\nabla a|^2 \, dx \, dt.
\]

- Space for ICs: \( P_0 := P \cap L^2 \)-closure \( \{ C_c^\infty \} \) to replicate \( u = 0 \) on \( \partial G \).

- **Existence, Energy Inequality**: \( u_0 \in P_0^\perp \). \( \exists u \in V \):
  - (W1), (W2)
  - continuous docking to IC: \( \| u(t, \cdot) - u_0 \|_{L^2(G)} \to 0 \) as \( t \to 0 \),
  - energy equality:

\[
\frac{d}{dt} \| u \|_{L^2} = 2 \| \nabla u \|_{L^2}.
\]

Equivalently for \( t > 0 \),

\[
\int_G |u(x,t)|^2 + \int_0^t \int_G |\nabla u(x,s)|^2 \, dx \, ds \leq \frac{1}{2} \int_G |u_0(x)|^2 \, dx.
\]