SPDEs

2/15/07

Lecture #6

(In the previous hw, compute expectations without squares)

- We have constructed Itô integrals for simple functions:

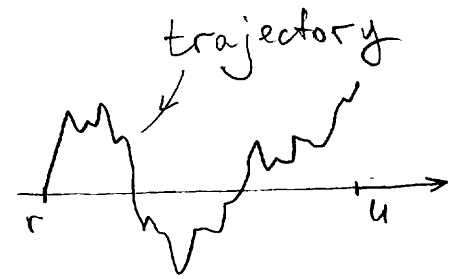
$$f(s) = \sum_{i=0}^{n} e_i \, \mathbb{1}_{(t_i, t_{i+1})},$$

where $e_i(w)$ is a r.v. which is $\mathcal{F}_{t_i}^W$-measurable.

Remember: ■ $\mathcal{F}_t^W = \sigma(W_s, s \le t)$.

If g is $\mathcal{F}_s^W$-meas. $\Leftrightarrow$ $g(W_s^0)$

where $W_u^r =$



E.g.: $*$ $g(s) = W(s) - W(s/2)$ is $\mathcal{F}_s^W$-meas.

$*$ $g(s+3)$ is NOT, because $s+3 > s$.

■ Sometimes it's convenient to consider COMPLETION of σ-algebras (even though all Wiener theory of B.m. would still work; but for other processes completion is necessary). All σ-algebras with respect to stoch. processes will be considered completed with respect to the given probability measure P on Ω.

Integral for simple functions on the interval $[0,T]$:

$$I(f) = \sum_{i=0}^{t_N=T} e_i \left(W_{t_{i+1}} - W_{t_i}\right)$$

<u>Properties</u>:

(a) $\mathbb{E}[I(f)] = 0$

(b) $\mathbb{E}|I(f)|^2 = \sum_{i=0}^{N-1} \mathbb{E} e_i^2 (t_{i+1} - t_i)$ This formula is the foundation of the whole theory!

(c) $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$

<u>Proof of (b)</u>: $|I(f)|^2 = \sum_i e_i^2 \left(W_{t_{i+1}} - W_{t_i}\right)^2$

$$+ \sum_{i \neq j} e_i e_j \left(W_{t_{i+1}} - W_{t_i}\right)\left(W_{t_{j+1}} - W_{t_j}\right)$$

$= A + B$.

$$\mathbb{E}\Big[\underset{\mathcal{F}^W_{t_i}\text{- meas}}{e_i^2} \underbrace{\left(W_{t_{i+1}} - W_{t_i}\right)^2}_{\sigma\left(W_{t_{i+1}} - W_{t_i}\right)}\Big] = \mathbb{E}\left[\mathbb{E}\left[e_i^2 \left(W_{t_{i+1}} - W_{t_i}\right)^2 \middle| \mathcal{F}^W_{t_i}\right]\right] =$$

by independence of $\mathcal{F}^W_{t_i}$ and $W_{t_{i+1}} - W_{t_i}$

$$= \mathbb{E}\left[e_i^2\, \mathbb{E}\left[\left(W_{t_{i+1}} - W_{t_i}\right)^2 \middle| \mathcal{F}^W_{t_i}\right]\right] = \mathbb{E}\left[e_i^2\, \mathbb{E}\left[\left(W_{t_{i+1}} - W_{t_i}\right)^2\right]\right]$$

$$= \mathbb{E}\left[e_i^2 (t_{i+1} - t_i)\right] =$$

$$= \mathbb{E}\left[e_i^2\right](t_{i+1} - t_i).$$

<u>Remark</u>: $\mathbb{E}[\xi \mid \{\emptyset, \Omega\}] = \mathbb{E}[\xi]$.

$$E[A] = \sum_i \mathbb{E}[e_i^2](t_{i+1} - t_i).$$

As far as B is concerned, assume $j > i$.

$$\mathbb{E}\left[e_i e_j (W_{t_{i+1}} - W_{t_i})(W_{t_{j+1}} - W_{t_j}) \right]$$
$$= \mathbb{E}\left\{ \mathbb{E}\left[e_i e_j (W_{t_{i+1}} - W_{t_i})(W_{t_{j+1}} - W_{t_j}) \mid \mathcal{F}^W_{t_j} \right] \right\}$$
$$= \mathbb{E}\left\{ e_i (W_{t_{i+1}} - W_{t_i}) e_j \underbrace{\mathbb{E}\left[(W_{t_{j+1}} - W_{t_j}) \mid \mathcal{F}^W_{t_j} \right]}_{0} \right\} = 0 \quad \blacksquare$$

Generalization: $f(t)$ – $\mathcal{F}^W_t$-measurable, with

$$\mathbb{E} \int_0^T f^2(t)\, dt < \infty.$$

$$\int_0^T f(s)\, dW_s = \lim_n \int_0^T f_n(s)\, dW_s$$

where f_n's are simple functions and

$$\mathbb{E}\left[\left| \int_0^T f(s)\, dW_s - \int_0^T f_n(s)\, dW_s \right|^2 \right] \xrightarrow{n \to \infty} 0.$$

$$\mathbb{E} \int_0^T |f(s) - f_n(s)|^2\, ds \longrightarrow 0,$$

Properties: (a) $\mathbb{E}\, I(f) = 0$

(b) $\mathbb{E}\left| \int_0^T f(s)\, dW_s \right|^2 = \int_0^T \mathbb{E}|f(s)|^2\, ds$, $\boxed{\text{Itô Isometry}}$

(c) I is linear

(d) $\int_0^t f(s)\, dW_s =: I_t(f)$.

There exists a version of $I_t(f)$ which is continuous f-n of t for all $\omega \in \Omega$; i.e. $\exists\, \tilde{\Omega} \subset \Omega$, s.t. $\mathbb{P}(\tilde{\Omega}) = 1$ and $I_t(f)$ is continuous on $\tilde{\Omega}$ (just redefine $I_t(f)$ on $\Omega \setminus \tilde{\Omega}$).

(e) $I_t(f)$ is a square-integrable martingale with respect to $\mathcal{F}_t^W$, i.e.

$$\mathbb{E}[\, I_t(f) \mid \mathcal{F}_s^W] = I_s(f).$$

Proof of (e): $$\underbrace{\sum_{i=0}^{n+m} e_i \left(W_{t_{i+1}} - W_{t_i} \right)}_{I_{n+m}} = \underbrace{\sum_{i=0}^{n} e_i \left(W_{t_{i+1}} - W_{t_i} \right)}_{I_n} + \underbrace{\sum_{i=n+1}^{n+m} e_i \left(W_{t_{i+1}} - W_{t_i} \right)}_{B}$$

$$\mathbb{E}\left[I_{n+m} \mid \mathcal{F}_{t_{n+1}}^W \right] = \sum_{i=0}^{n} e_i \left(W_{t_{i+1}} - W_{t_i} \right) + 0.$$

For generic $f(s)$, consider seq. of simple functions. ∎

Stratonovich Integral.

Approximate B.m. with sequence $W^n \longrightarrow W$, with W^n smooth.

$$\frac{dX_t^n}{dt} = b(t, X_t^n) + \sigma(t, X_t^n)\frac{dW_t^n}{dt}$$

also written as: $dX_t^n = b(t, X_t^n)\,dt + \sigma(t, X_t^n)\,dW_t^n$

or: $X_t^n = X_0^n + \int_0^t b(s, X_s^n)\,ds + \int_0^t \sigma(s, X_s^n)\,dW_s^n$.

$$\Rightarrow \qquad X_0 + \int_0^t b(s, X_s)\,ds + \underbrace{\int_0^t \sigma(s, X_s)\,dW_s + \frac{1}{2}\int_0^t (\sigma \cdot \sigma_x)(s, X_s)\,ds}_{\int_0^t \sigma(s, X_s) \circ dW_s}$$

Stratanovich Integral construction:

$$S(f) = \sum_i e_i \left(W_{t_{i+1}} - W_{t_i}\right).$$

where e_i is $\mathcal{F}^W_{\frac{t_{i+1} - t_i}{2}}$ //